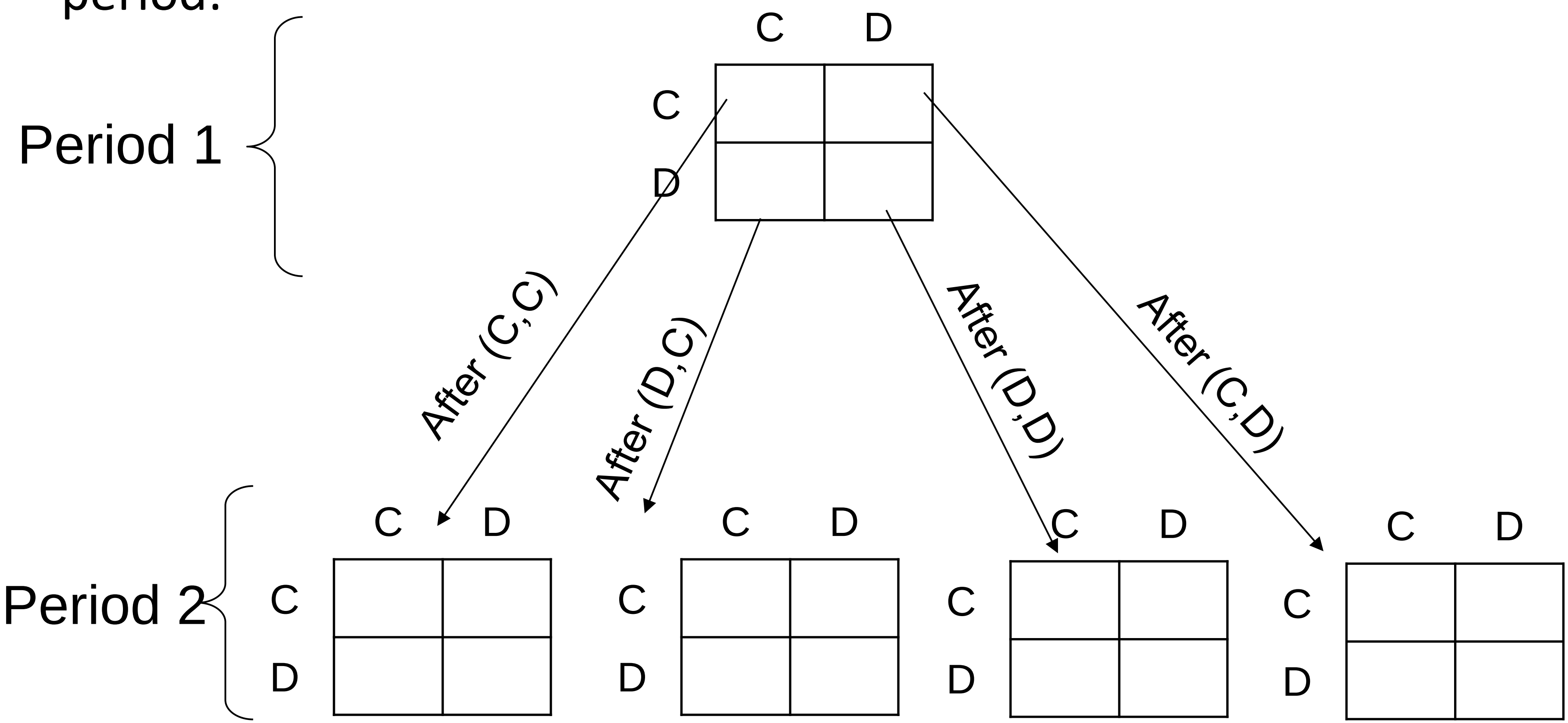
Partial Lecture Notes

REPEATED GAMES

Repeated games

A series of repeated simultaneous move games is really a large extensive form game that allows for simultaneous moves each period:



Repeated games

- A repeated game is sequential move game constructed from a (simultaneous move) base game. The base game is called a stage game (e.g., PD)

- Any stage game can be repeated (not just the PD). We will study PD's here.

- Games can be repeated a finite or an infinite number of times. This matters.

Repeated games

Length of Repetition

- Finite horizon ($T < \infty$)
  - Solve by backward induction
- Infinite horizon ($T = \infty$)
  - Cannot be solved by backward induction (since there is no end)

Goals of the analysis

- Does cooperation emerge if we repeat the PD? If so, under what conditions?
- What are the equilibria in a repeated PD?
- How do we analyze infinitely repeated games?
- Are there general results about repeated games?

Time preferences

Time preferences

- Key assumption: in many settings a payoff in the future is worth less than today.
- **Discount factor** $\delta \in (0, 1)$ parameterizes *patience*.
- Utility (present value at time t) of receiving X at time t+1 is δX.

- Suppose the *interest rate* is r.

  If you invest X in period t, then you want to get a bigger return in t+1. Typically the amount returned in t+1 is

  $X(1+r) = X + \underline{Xr}$, where $0 < r \leq 1$.

  principle interest interest rate

Time preferences

Time preferences

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- **Discount factor** $\delta \in (0, 1)$ parameterizes *patience*
- Utility (present value at time t) of receiving X at time t+1 is δX.

- Suppose the *interest rate* is r.

  If you invest X in period t, then you want to get a bigger return in t+1. Typically the amount returned in t+1 is $X(1+r) = X + Xr$, where $0 < r \leq 1$.
- Easier: the present value of receiving X tomorrow is less than it is today, so we have to discount X tomorrow compared with X today (i.e. use δX for t+1).

Time preferences

Consider four periods of {C,C} in this PD ->

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

Period (t)		1		2		3		4
payoff	=	*3*	+	*δ3*	+	*δ(δ3)*	+	*δ(δδ3)*

$$= \quad 3 + \delta^1 3 + \delta^2 3 + \delta^3 3$$

$$= \quad \sum_{t=1}^{4} \delta^{t-1} 3$$

Time preferences

Consider four periods of {C,C} in this PD ->

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

Period (t) 1 2 3 4

payoff = $3 + \delta 3 + \delta(\delta 3) + \delta(\delta\delta 3)$

$$= 3 + \delta^1 3 + \delta^2 3 + \delta^3 3$$

$$= \sum_{t=1}^{4} \delta^{t-1} 3$$

$$U_i(h) = \sum_{t=1}^{T} \delta^{t-1} u_i(x_t)$$

This is a general formula for finite repetition.

Time preferences

Discounted sum of payoffs (total net present value)

$$U_i(h) = \sum_{t=1}^{T} \delta^{t-1} u_i(x_t)$$

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where $u_i(x)$ is individual i's utility for outcome x in period t.

(Different periods may have different outcomes).

Time preferences

- Practice

Stage game

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

1. What is the discounted utility for player 1 (row) in a 3 period repeat of the stage game above with play (D,D), (C,C), (D,C)? [hint: use δ].

2. What is the discounted utility for player 2 (column) in the same game from the same play?

Time preferences

- Practice

Stage game

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

3. What is the discounted utility for player 1 (row) in a 20 period repeat of the stage game above with play (D,D), (C,C), followed by (D,C) for 18 rounds? [hint: use ∑ and δ].

Infinitely repeated game

Maybe we can engender cooperation if the game is played an infinite number of periods.

- After all, it was the last period that made defection rational and caused the game to unravel.

Infinitely repeated game

What is the equilibrium (or equilibria) in an *infinitely* repeated PD?

- $T = \infty$
- e.g., h = ((C,C), (C,D), (C,D), (C,D), …)
- Payoffs are the sum of an infinite series $\rightarrow \infty$

- The discount factor can be interpreted as
  - Impatience (how much you are willing to wait for a payoff).
  - The probability the game ends.

Geometric Progression

Consider a *constant* payoff of c for T finite periods:

$$S_T = \sum_{t=1}^{T} \delta^{t-1} c = c\left(1 + \delta + \delta^2 + \ldots + \delta^{T-1}\right)$$

We now use a trick to simplify the above equation. Note…

$$\delta S_T = c\delta\left(1 + \delta + \delta^2 + \delta^3 + \ldots + \delta^{T-1}\right)$$

$$S_T - \delta S_T = c\left(1 + \delta \ \ + \delta^2 + \ldots + \delta^{T-1}\right) - c\left(\delta + \delta^2 + \delta^3 + \ldots + \delta^T\right)$$

$$S_T = \frac{c(1-\delta^T)}{1-\delta}$$

For infinite periods: As T → ∞,

$\delta^T \to 0$ and for T = ∞

$S_T = c / (1 - \delta)$.

Time preferences

Discounted sum of streams of constant payoff c:

$$\sum_{t=1}^{T} \delta^{t-1} c = \frac{c(1+\delta^{T})}{1-\delta} \qquad \sum_{t=1}^{\infty} \delta^{t-1} c = \frac{c}{1-\delta}$$

Mathematically, this is a **geometric series**, so discounting each future period by a constant discount factor of δ is called **geometric discounting**.

Cardinality matters (just like it did for expected utility)

Strategies

- A strategy specifies an action for *every* period of the game.
- In an infinitely repeated game, the set of strategies is infinite.
- We will restrict attention to a few strategies that are easy to describe:
  - Always defect – D in every period.
  - Always cooperate – C in every period.
  - Grim trigger: cooperate in first period, defect forever if other player has defected in a previous period.
  - Tit-for-tat: cooperate in first period, copy other player's action in next period.

Nash equilibrium and SPE

- Sequential Equililbrium
  - How does one apply backward induction to a game that has no end?
  - Answer: you don't. Hence you would study sequential equilibria (i.e. sub-game perfect equilibria) differently.

- We will focus on Nash equilibrium
  - Because analyzing sub-game perfect equilibria in repeated games does not give us any additional insights. Furthermore, N.E. are much easier.

- Nash equilibrium
  - Set of strategies such that no player has an incentive to deviate
  - Check for deviations from something we suspect is Nash.

Always defect

Stage game

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

- Assume common discount factor δ
  - Player 1: D, D, D, …
  - Player 2: D, D, D, …
  - Payoffs 1: 1, 1δ, 1δ^2, …
  - Payoffs 2; 1, 1δ, 1δ^2, …
  - Sum of payoffs: c / (1 - δ) = 1 / (1 - δ).
  - This is a NE because there is no incentive to unilaterally deviate to another (repeated) strategy.
    - Note: any deviation from (all D, all D) leads to a lower payoff in the deviating period. Hence, (all D, all D) is a NE.

Grim trigger (GT)

Stage game

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

C in first period.
C as long as other plays C.
D forever if other plays D in any round.

- Assume common discount factor δ
  - Player 1: C, C, C, …
  - Player 2: C, C, C, …
  - Payoffs 1: $3, 3\delta, 3\delta^2, \ldots$
  - Payoffs 2; $3, 3\delta, 3\delta^2, \ldots$
  - Sum of payoffs: $c / (1 - \delta) = 3 / (1 - \delta)$.
  - Note: if player 1 deviates to "always D" (or identically grim trigger with D in the first round), then the two will get:
    - Player 1: D, D, D, …
    - Player 2: C, D, D, …

Grim trigger (GT)

Stage game

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

C in first period.
C for any history such that no player has ever played D.
D if either player has ever played D.

- **It is rational for player 1 to deviate to "always D" iff:**

EU_1(always D) > EU_1(GT, GT)

$$5 + 1(\delta + \delta^2 + \delta^3 + ...) > \frac{3}{(1-\delta)}$$

$$5 + \delta(1 + \delta + \delta^2 + ...) > \frac{3}{(1-\delta)}$$

$$5 + \sum_{t=1}^{\infty} \delta^{t-1} > \frac{3}{(1-\delta)}$$

$$5 + \frac{\delta}{(1-\delta)} > \frac{3}{(1-\delta)}$$

$$5 - 5\delta + \delta > 3$$

$$2 > 4\delta$$

$$\delta < 1/2$$

If δ < ½, then this deviation (and other deviations) are rational.

If δ ≥ ½, then (GT, GT) is a Nash Equilibrium, generating the outcome (C,C) in every period.

Note: deviating to "always defect," in a later period produces the same condition. See attached.

Always cooperate

Stage game

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

C all periods.

- Assume common discount factor δ
  - Player 1: C, C, C, …
  - Player 2: C, C, C, …
  - Payoffs 1: 3, 3δ, $3δ^2$, …
  - Payoffs 2; 3, 3δ, $3δ^2$, …
  - Sum of payoffs: c / (1 - δ) = 3 / (1 - δ).
    - Note: if player 1 deviates to "always D," then he will get 5 / (1 - δ).
    - This deviation is rational if 5 / (1 - δ) > 3 / (1 - δ), which is true for all δ.
  - Hence, {always C; always C} is not a N.E.

Intuition

- If players are *sufficiently patient*, then cooperation (C,C) on the path of play is supported by a Nash equilibrium where both players use the Grim trigger strategy

- If players are impatient, then cooperation cannot be sustained in equilibrium

- Cooperation requires
  - *Threat of future punishment* for not cooperating must exist.
  - Infinite horizon.
  - Players must be *sufficiently patient* (long-term gain from cooperating must exceed short-term gain from defecting minus long-term cost of defecting)

Steps in analysis

1. Determine the play implied by the strategies.
2. Compute discounted sum of payoffs.
3. Find best possible deviation for one player (usually all defect, or defect in first period). If this one outperforms, then you don't have an equilibrium.
4. Set up the Nash equilibrium condition (inequality)
5. Solve to determine if there is a feasible value of δ (between 0 and 1), where equilibrium can be sustained.

Tit for tat

Stage game

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

- Start with C
- Play C if other player played C in previous period
- Play D if other player played D in previous period

U_1(TFT,TFT) = ?

Practice: Do first two previous steps on this PD (new payoffs).

Tit for tat

Stage game

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

- Start with C
- Play C if other player played C in previous period
- Play D if other player played D in previous period

U_1(TFT,TFT) = $\frac{2}{(1-\delta)}$

Practice: Do first two previous steps on this PD (new payoffs).

Step 1:

- Player 1: C, C, C, …
- Player 2: C, C, C, …

Step 2:

- Payoff 1: $3 + 3\delta + 3\delta^2 + \ldots = \frac{2}{(1-\delta)}$

Tit for tat

Stage game

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

- Start with C
- Play C if other player played C in previous period
- Play D if other player played D in previous period

U_1(TFT,TFT) = $\frac{2}{(1-\delta)}$

Practice: Do third step on this PD.

Deviation 1: Always defect

Player 1: D, D, D, …

Player 2: C, D, D, …

Payoff 1: $3 + \delta + \delta^2 + \delta^3 \ldots$

Tit for tat

Deviation 1: Always defect.

Payoff 1: $3 + (\delta + \delta^2 + \delta^3 \ldots)$

$3 + \delta(1 + \delta + \delta^2 \ldots)$

$$3 + \delta \sum_{t=1}^{\infty} \delta^{t-1}$$

$$3 + \frac{\delta}{1 - \delta}$$

Tit for tat

Stage game

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

$$U_1(TFT, TFT) = \frac{2}{1-\delta}$$

- Start with C
- Play C if other player played C in previous period
- Play D if other player played D in previous period

Deviation 1: Always defect

This is a rational deviation iff:

$$3 + \frac{\delta}{(1-\delta)} > \frac{2}{(1-\delta)}$$

$$(3 - 3\delta) + \delta > 2$$

$$3 - 2\delta > 2$$

$$\delta < 1/2$$

If the players value the future moderately, $\delta > ½$, cooperation can be sustained between these strategies.

If the players don't value the future moderately, $\delta < ½$, cooperation cannot be sustained between these strategies.

Tit for tat

Stage game

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

$U_1(\text{TFT,TFT}) = \frac{2}{(1-\delta)}$

Practice:

- Start with C
- Play C if other player played C in previous period
- Play D if other player played D in previous period

Why is looking at deviation in the first round sufficient for the case of TFT against TFT?

Cooperation in infinitely repeated PD

- Cooperation along the equilibrium path of play can be supported by several different strategy profiles.
- Cooperation is supported by the threat of punishment and a sufficient level of patience.
  - Note: (all C, all C) is not an equilibrium strategy. Even a nice strategy must be able to punish.
- The level of patience required is smaller if punishment is more severe (e.g., grim trigger requires less patience, TFT requires more patience).

Alternate equilibrium path

- Instead of (C,C) in every period, is there a NE where the players alternate between (D,C) and (C,D)?
- Consider an alternating grim trigger set of strategies (*AltGT*):
  - Player (D,C) in odd number periods, play (C,D) in even number periods
  - If either player deviates from this path of play, play D forever

Stage game

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

$$U_1(AltGT, AltGT) = 3 + 0\delta + 3\delta^2 + 0\delta^3 + \ldots$$
$$= \frac{3}{1-\delta^2}$$

$$U_2(AltGT, AltGT) = 0 + 3\delta + 0\delta^2 + 3\delta^3 + \ldots$$
$$= \frac{3\delta}{1-\delta^2}$$

Alternate equilibrium path

Since Player 1 gets highest payoff in period 1, deviate to D in period 2

$$U_1(Dev, AltGT) = 3 + \delta + \delta^2 + \delta^3 + \ldots$$
$$= 3 + \delta + \frac{\delta^2}{1-\delta} = 3 + \frac{\delta}{1-\delta}$$

Player 1 has no incentive to deviate if

$$U_1(AltGT, AltGT) \geq U_1(Dev, AltGT)$$
$$\frac{3}{1-\delta^2} \geq 3 + \frac{\delta}{1-\delta}$$
$$\delta \geq \frac{1}{2}$$

Alternate equilibrium path

Since Player 2's best deviation is to start playing D in period 1

$$U_2(Dev, AltGT) = 1 + \delta + \delta^2 + \delta^3 + \dots$$
$$= \frac{1}{1-\delta}$$

Player 2 has no incentive to deviate if

$$U_2(AltGT, AltGT) \geq U_2(Dev, AltGT)$$
$$\frac{3\delta}{1-\delta^2} \geq \frac{1}{1-\delta}$$
$$\delta \geq \frac{1}{2}$$

Alternate equilibrium paths

- Thus, for the stage game with the payoffs given, there is a Nash equilibrium where players alternative between (D,C) and (C,D) along the equilibrium path.
- This suggests that outcomes other than full cooperation can be supported in equilibrium.

Remarks

The folk theorem (which we did not introduce) tells us that in infinitely repeated games there is a *multiplicity of equilibria* – we cannot make sharp empirical predictions.

In the PD, cooperation is sustainable in equilibrium—but it is *not the only possible outcome*. All defect is in equilibrium against all defect as well.

The folk theorem tells us which *payoffs* are supportable in *some* Nash equilibrium. It does *not* tell us anything the actual strategy profiles that might be used.