THEORY: SUBGAME PERFECT EQUILIBRIUM

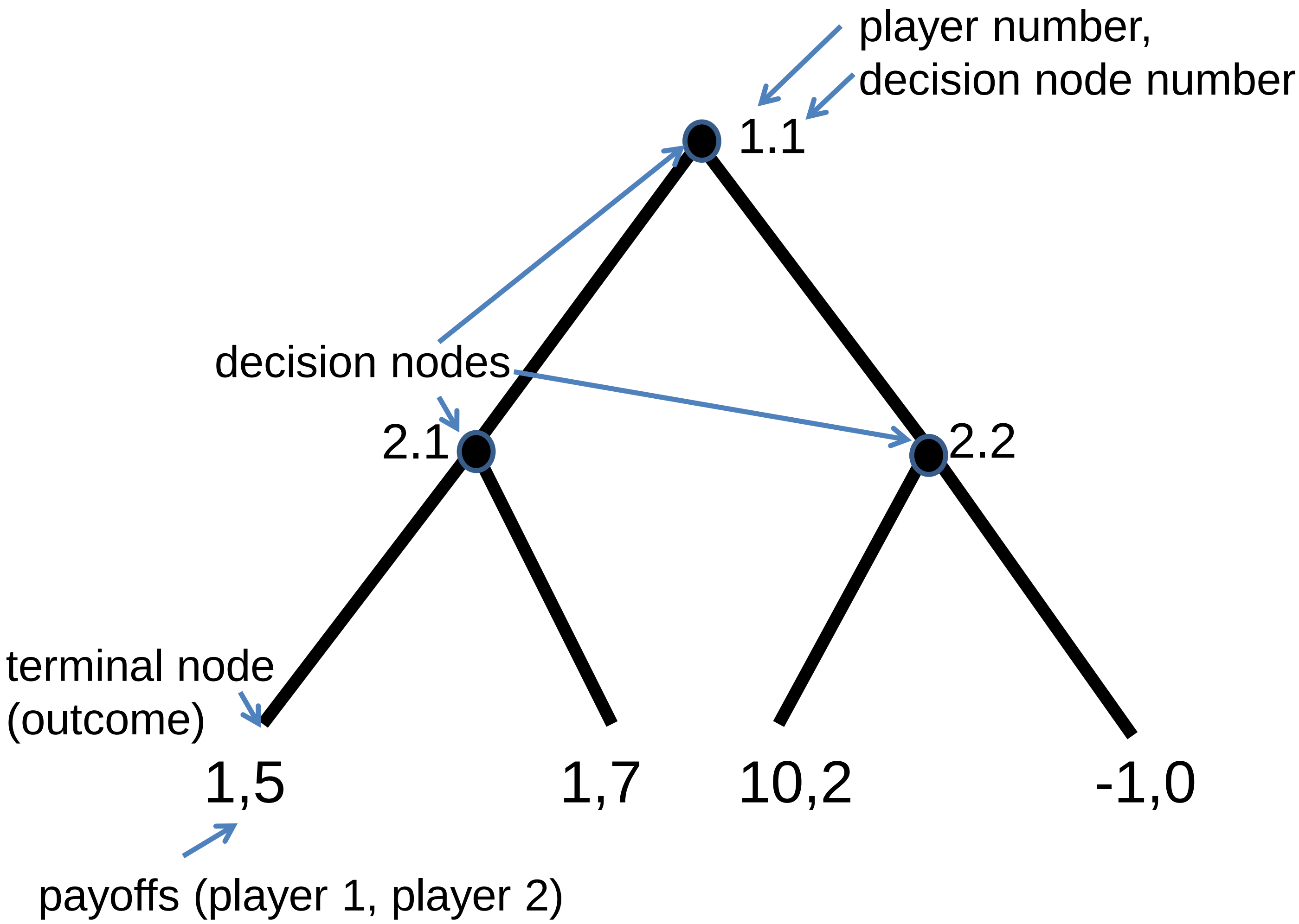
Extensive Form Games

- Strategic (or normal) Form Games
  - Time is absent

- Extensive Form Games
  - Capture time
  - With the introduction of time, players can adopt strategies contingent on the moves of others.

- Key ideas
  - Game trees: graphical representations
  - Histories: sequences of moves
  - Strategies: complete plans of actions
  - Subgame Perfect Equilbrium: strengthens (refines) Nash equilibrium concept

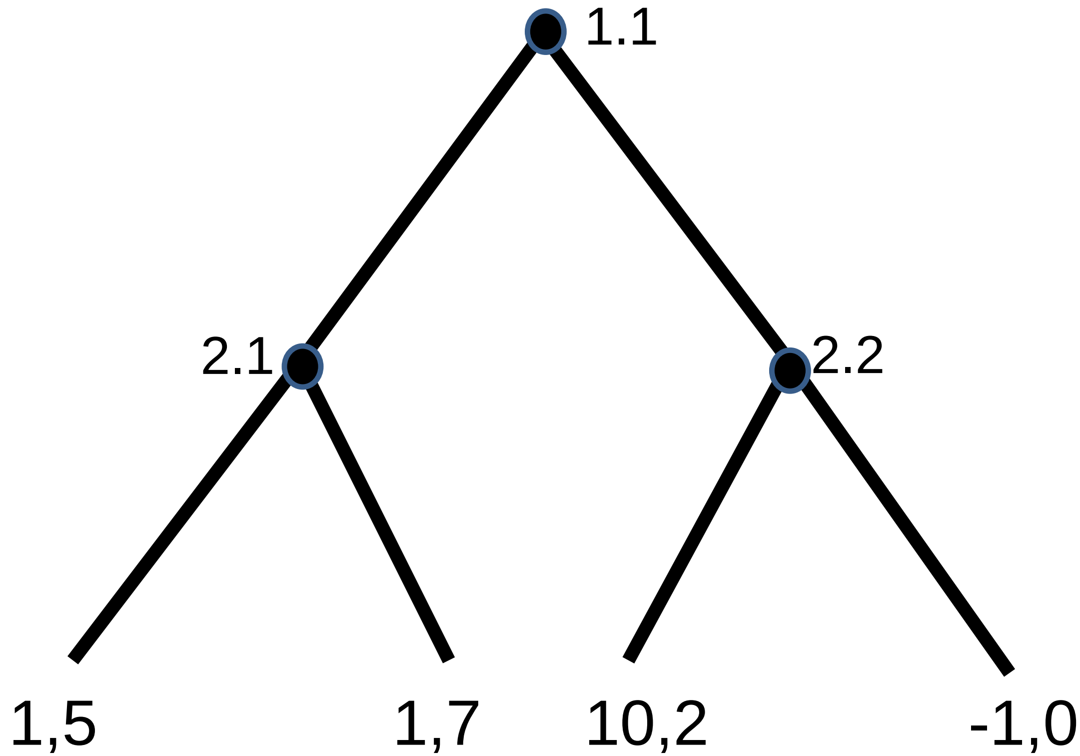
Extensive Form Games



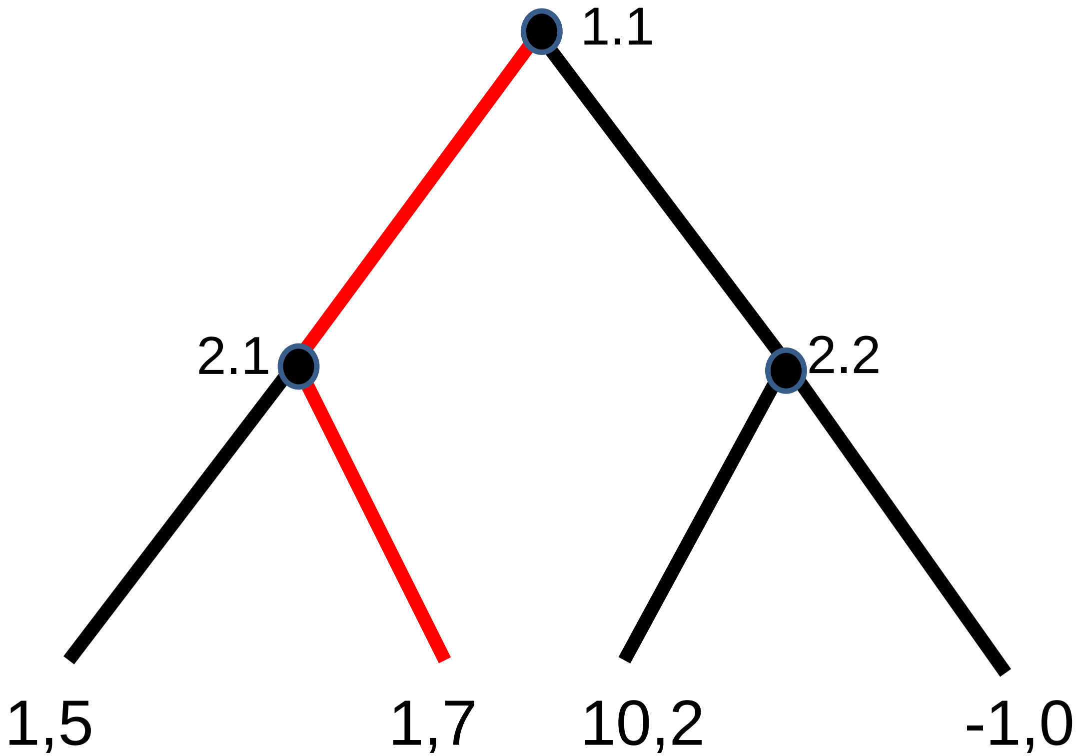
Extensive Form Games

- Histories
  - A particular sequence of moves that occurs in a game (e.g., a_1, a_2, …, a_k) is called a **history**.
  - A **subhistory** of this history is either the empty history (the start of the game) or a sequence with the property that (a_1, a_2, …, a_m) for $m \leq k$.

Extensive Form Games



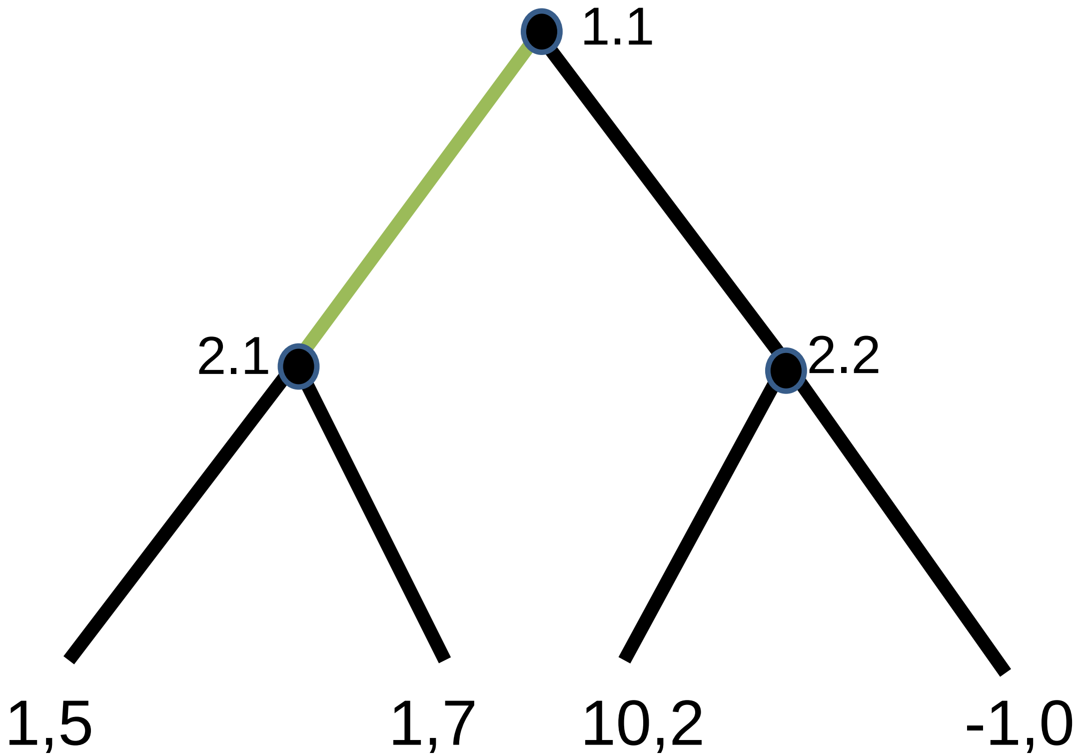
Extensive Form Games



Here’s a history.

How many histories does this game have?

Extensive Form Games



Here's a subhistory.

Note: after this subhistory we know we are at 2.1

Extensive Form Games

Definition

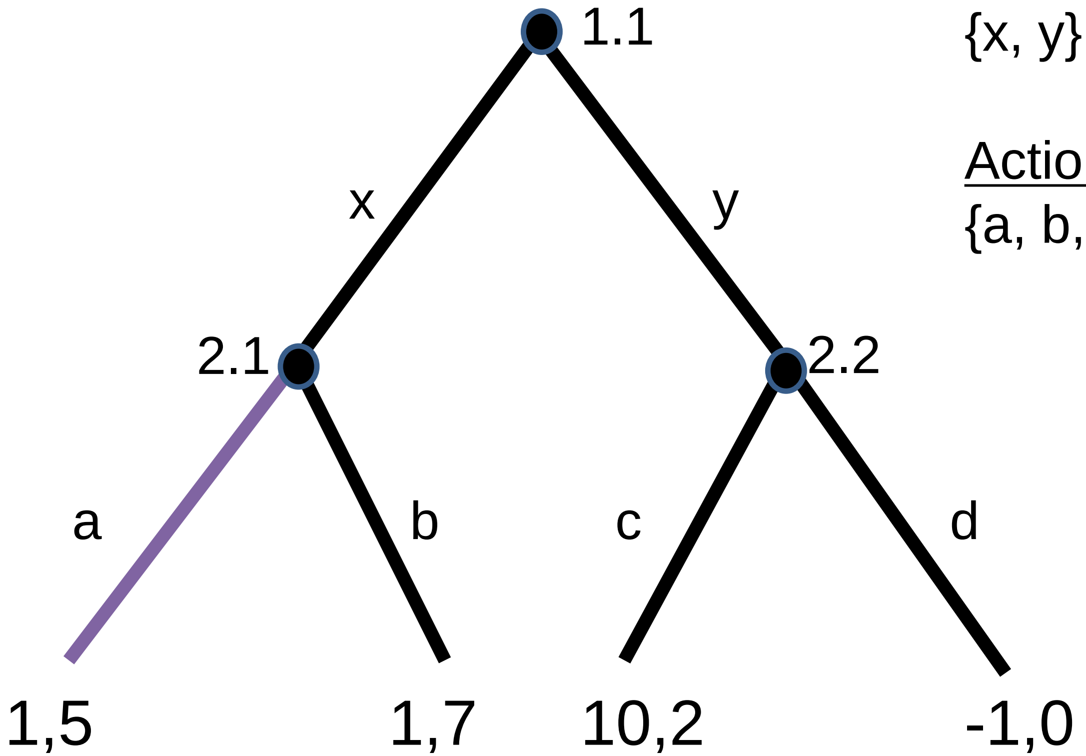
- An extensive form game contains the following elements.
  - A set of players.
  - A set of terminal histories.
  - An assignment of a player to each decision node.
  - Preferences, for each player, over the set of terminal histories.

Extensive Form Games

Strategies

- Definition.
  - A strategy for player *i* in an extensive form game assigns an action to each subhistory at which it is *i*'s turn to move.
  - Note: this definition implies there is a distinction between strategy and action.
    - A **strategy** is a complete plan of action for the entire game. A strategy must specify an action for the player to take at each subhistory where a player would potentially move, even if these subhistories are never attained.
    - An **action** is a move at a particular decision node.

Extensive Form Games

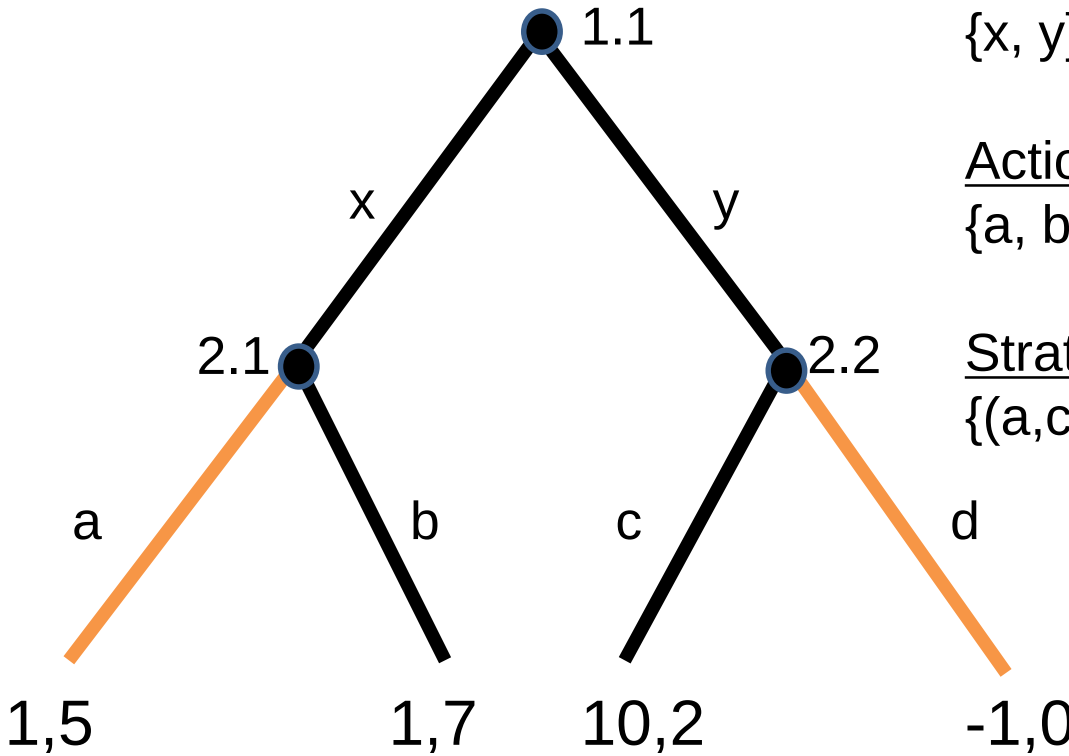


Actions for player 1:
{x, y}.

Actions for player 2:
{a, b, c, d}.

Here’s an action for player 2 at decision node 2.1: play a.

Extensive Form Games



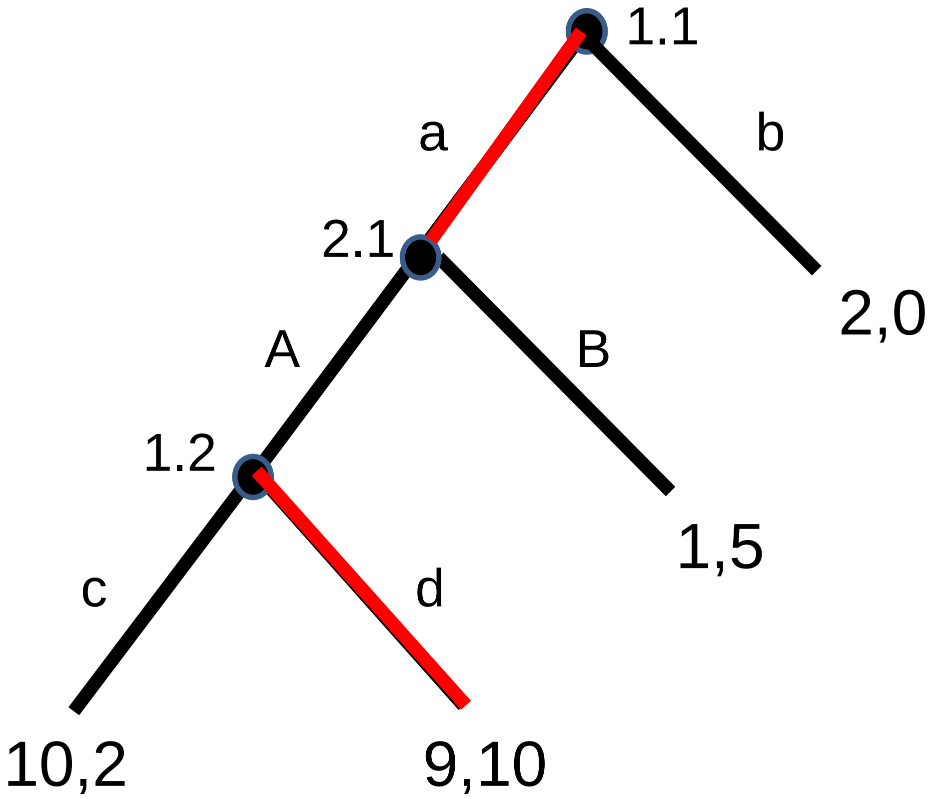
Actions for player 1:
{x, y}.

Actions for player 2:
{a, b, c, d}.

Strategies for player 2:
{(a,c),(a,d),(b,c),(b,d)}.

Here's a strategy for player 2: (a,d).

Extensive Form Games



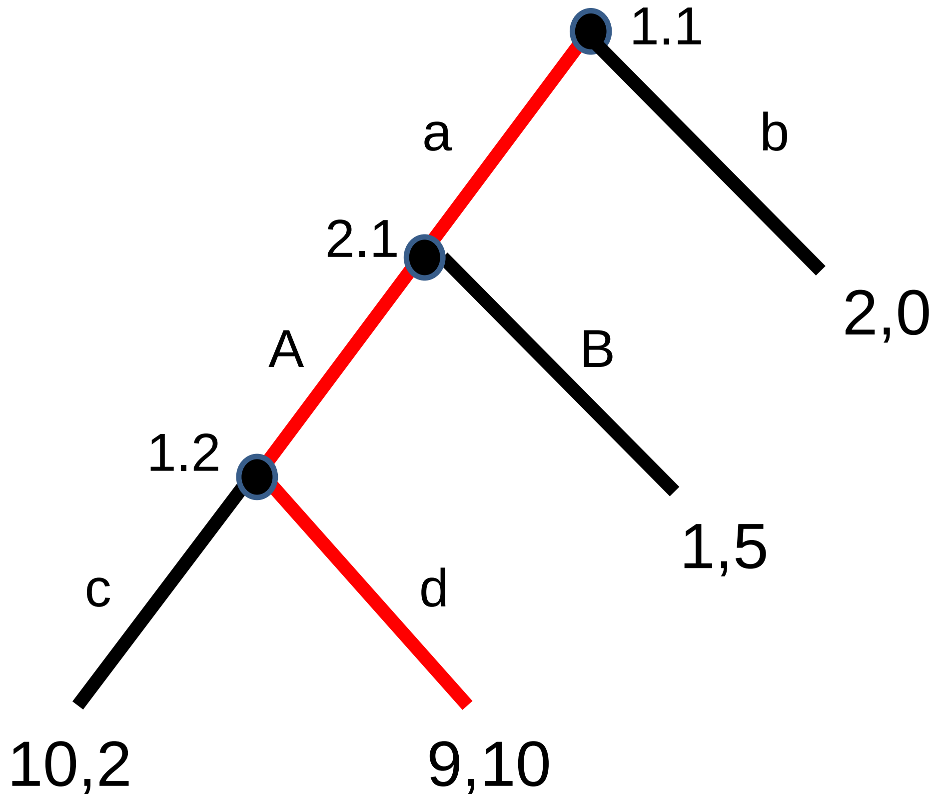
One strategy for player 1 is: (a, d).

Extensive Form Games

Outcomes

- A strategy profile (i.e. strategies for all players) produces a path of play through the tree. The terminal node or outcome that is reached under strategy profile *s* is denoted by O(*s*).

Strategy Profile



s = {(a,d); A} leads to the outcome with payoff (9,10).

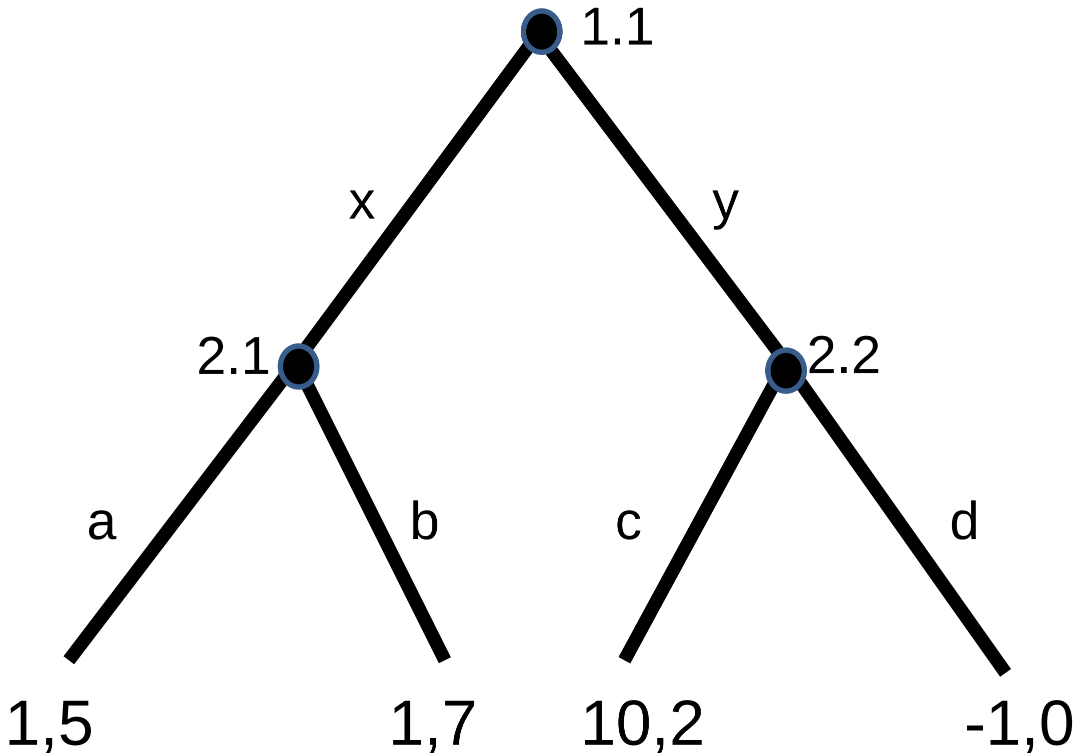
Extensive Form Games

Nash Equilibrium

- A strategy profile s* is a Nash Equilibrium of an extensive form game if and only if

  $$\forall i \in N, U_i(O(s_i^*, s_{-i}^*) \geq U_i(O(\tilde{s}_i, s_{-i}^*)) \text{ for all } \tilde{s}_i \in S_i.$$

Nash Equilibrium



Convert to normal form…

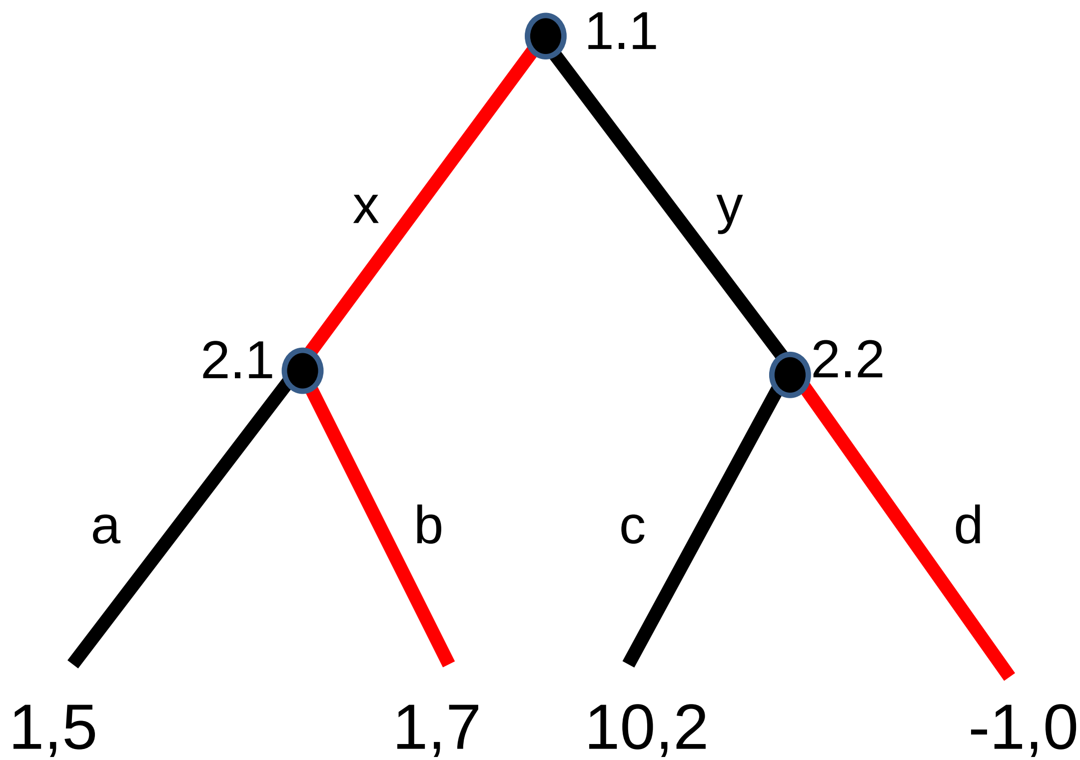
Nash Equilibrium

	(a,c)	(a,d)	(b,c)	(b,d)
x	1,5	1,5	1,7	1,7
y	10,2	-1,0	10,2	-1,0

What are the Nash Equilibria?
{x;(b,d)}, {y;(a,c)}, {y;(b,c)}

But some of these equilibria seem less credible than others, because with sequencing it is not rational to carry out what is promised.

Nash Equilibrium



For example: s = {x;(b,d)} is not credible because player 2 would not play d at 2.2

Backwards Induction

Equilibrium Refinement

- Backwards induction identifies Nash Equilibria with credible threats and credible promises. This motivates our next equilibrium concept.

Subgame Perfect Equilibrium

- Subgame Perfect Equilibrium requires that players play a Nash Equlibrium in every subgame of the game.
  - As a result, every subgame perfect equilibrium is a Nash equlibrium, but not the other way around.

Subgames

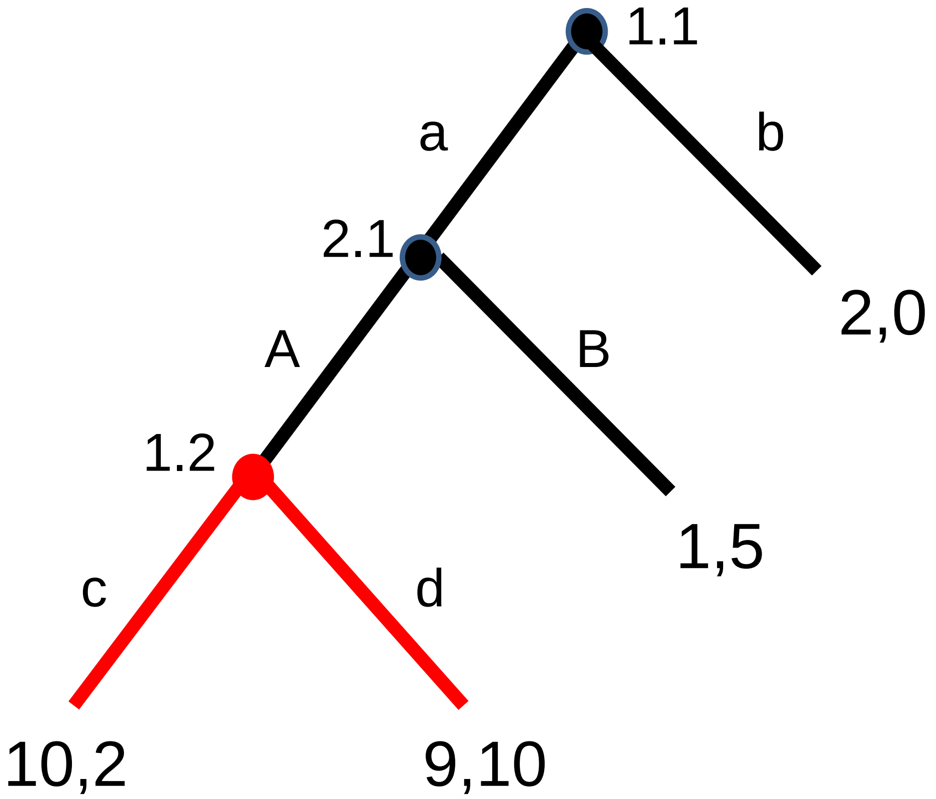
Subgames

- A subgame begins at a particular decision node, and contains the rest of the game from that node forward. The entire game itself is also a subgame.

Formal Definition

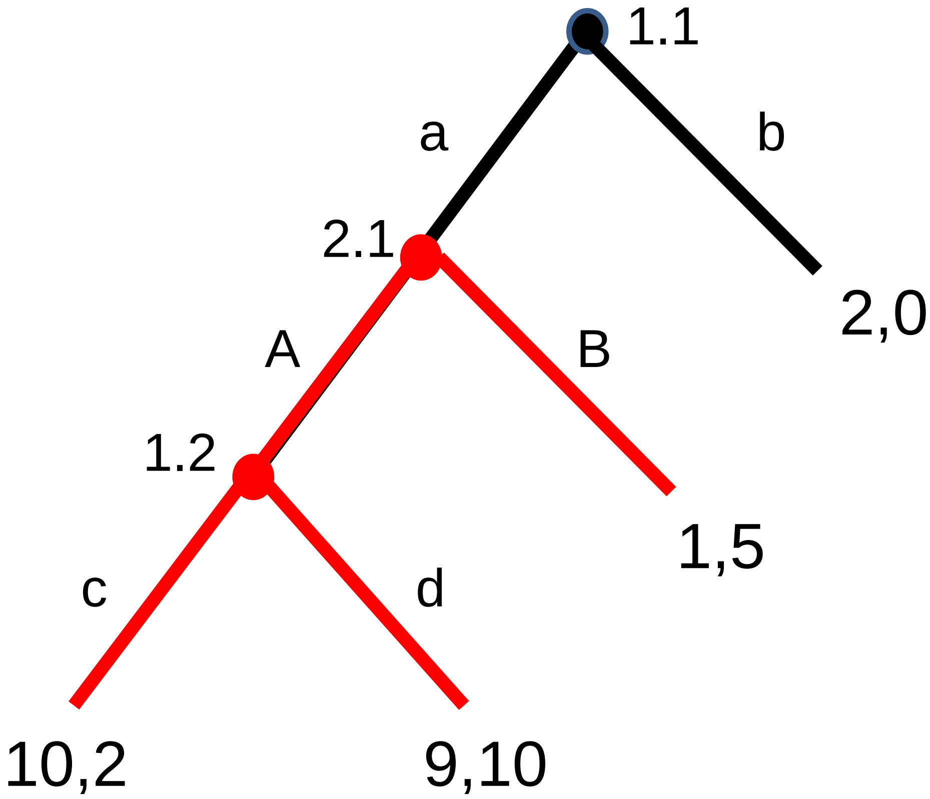
- The subgame of game G that follows history h is the following game G(h).
  - The set of players is equal to the set of players in G.
  - The set of terminal histories are the sequences h' such that (h,h') is a terminal history of G.
  - The player function assigns the player P(h,h'') to each proper subhistory h'' of h'.
  - The preferences of players over terminal histories are as in G.

Strategy Profile



Here’s a subgame.

Strategy Profile



Here’s another subgame.
The only other subgame is the entire tree.

Subgame Perfect Equilibrium

Definition

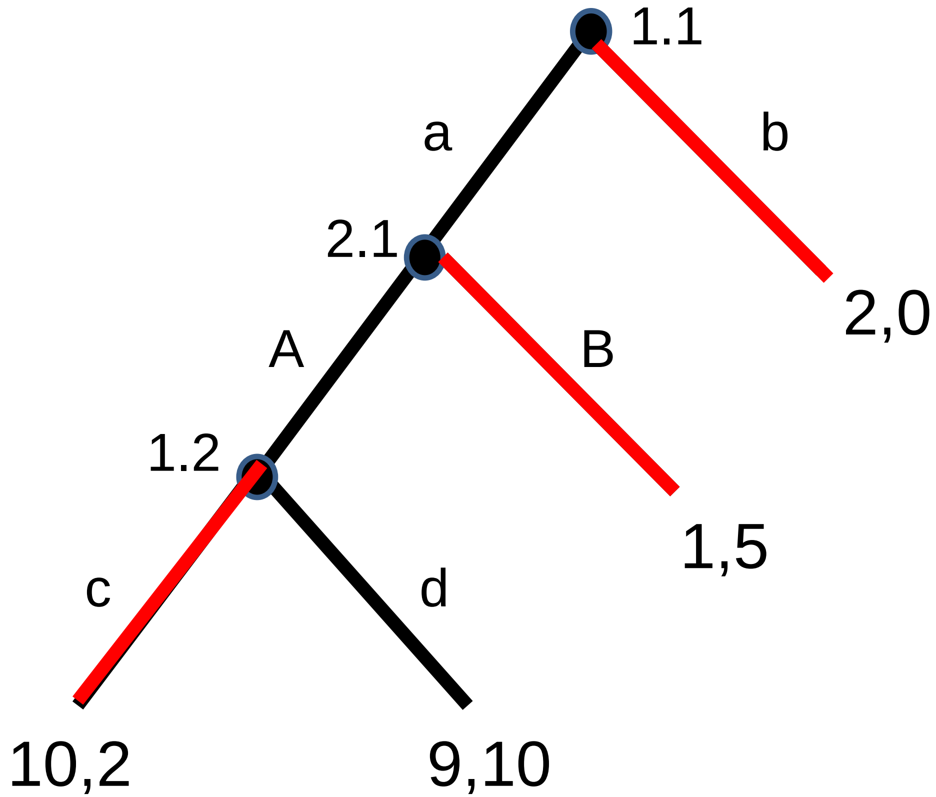
- A strategy profile s* is a **subgame perfect equilibrium** if and only if $\forall i \in N$, and for all histories *h* after which it is *i*'s turn to move,

$$U_i(O_h(s_i^*, s_{-i}^*) \geq U_i(O_h(\tilde{s}_i, s_{-i}^*)) \text{ for all } \tilde{s}_i \in S_i.$$

Theorem

- In games with perfect information and finite actions, backwards induction identifies the set of subgame-perfect equilibria of the game exactly.

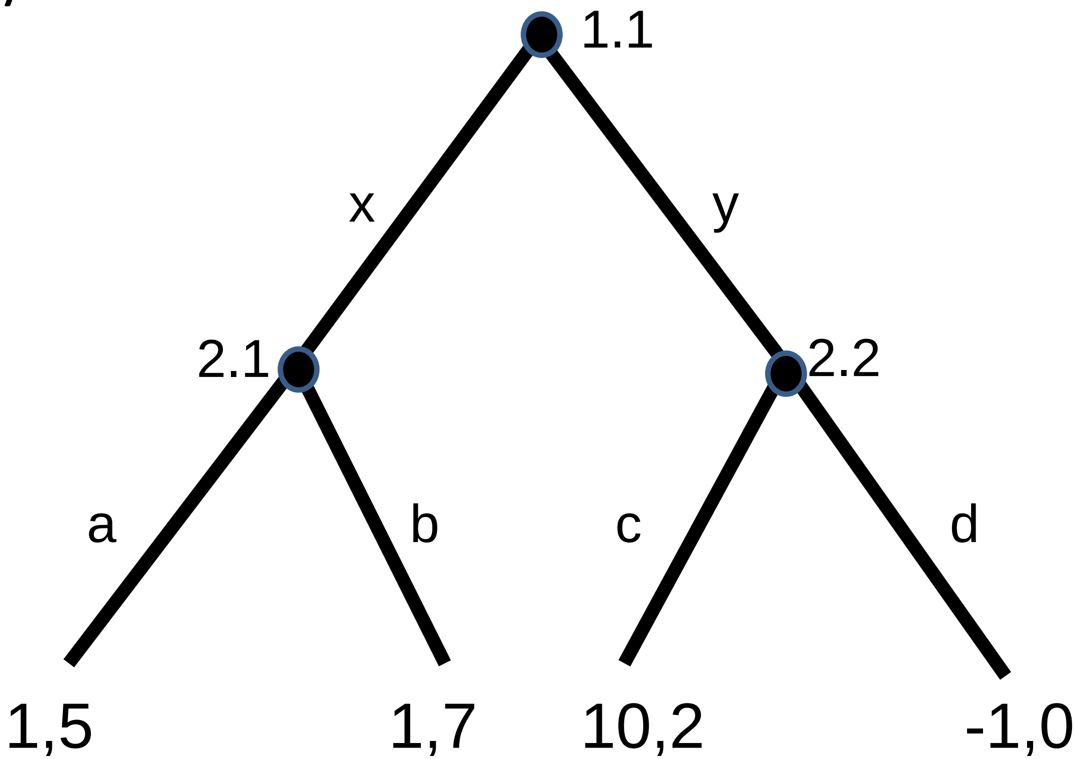
Subgame Perfect Equilibrium



Working backward…
SPE = {(b,c);B}

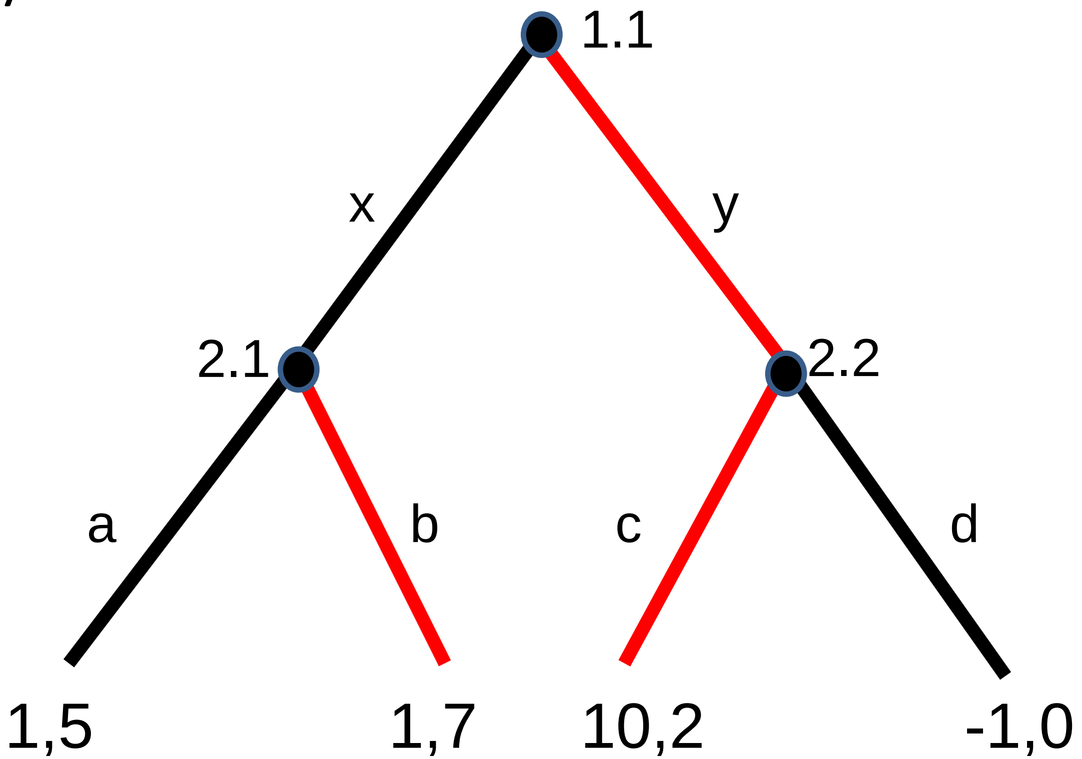
Subgame Perfect Equilibrium

- You try



Subgame Perfect Equilibrium

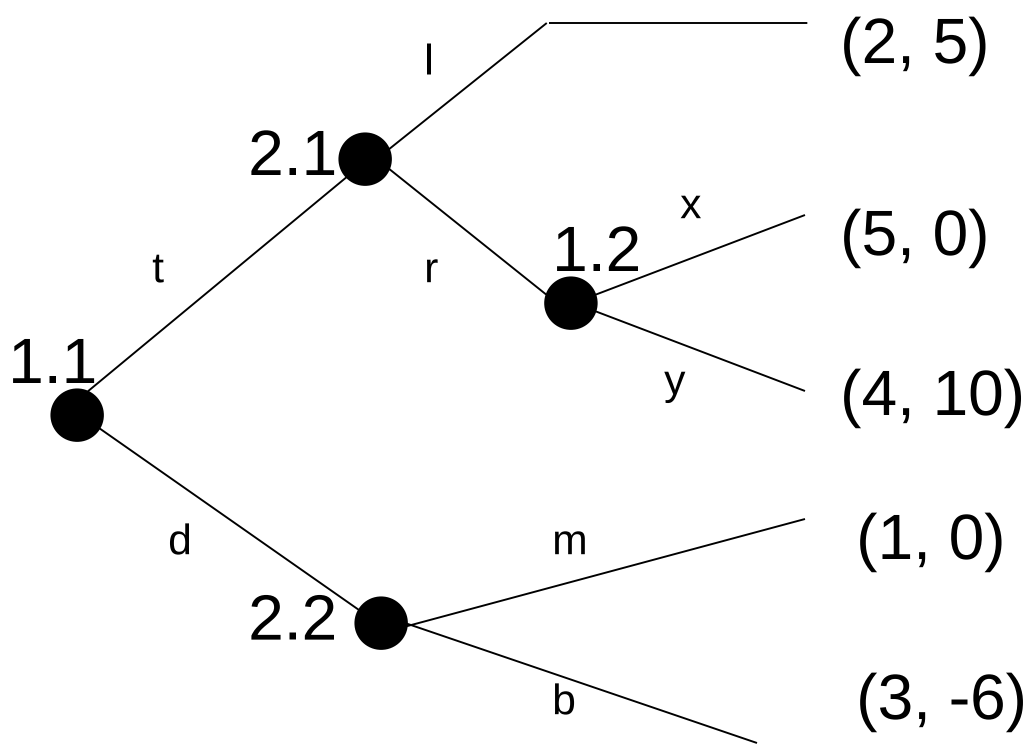
- You try



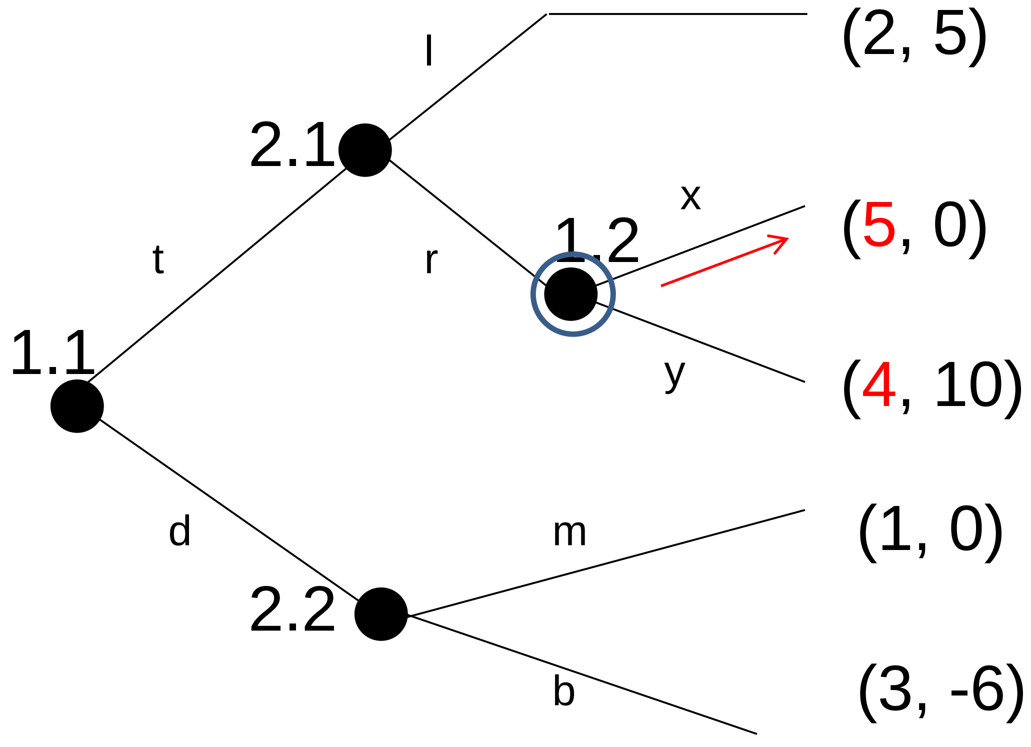
SPE = {y;(b,c)}
Note, the SPE is only one of the three Nash Equilibria.

Subgame Perfect Equilibrium

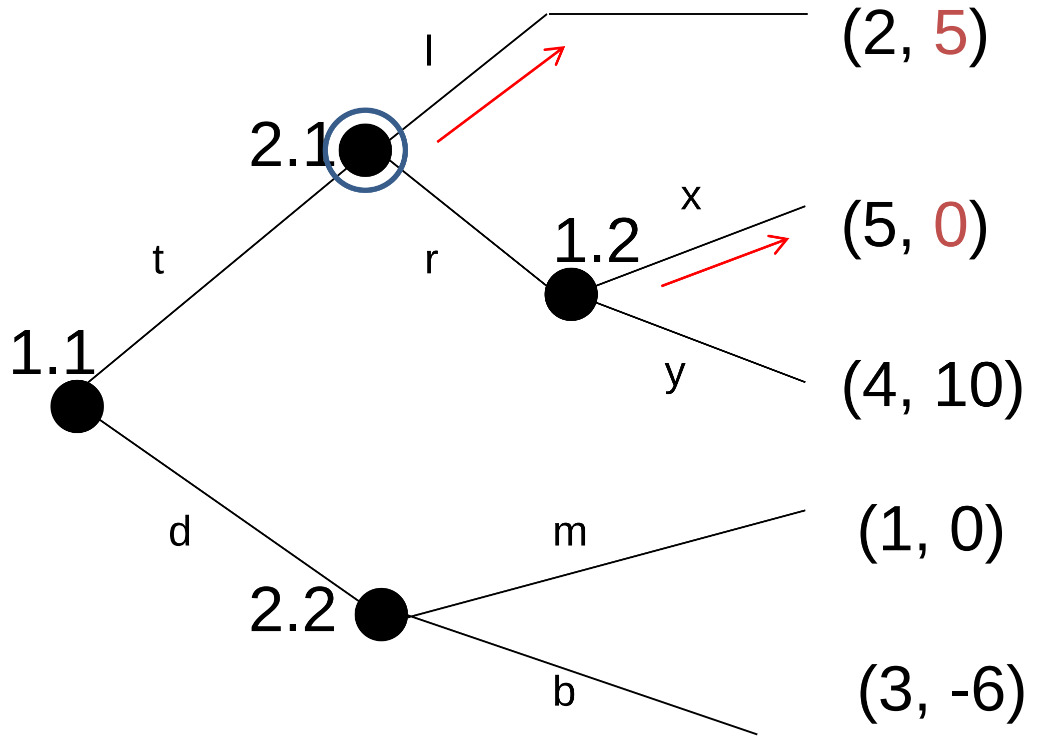
- You try



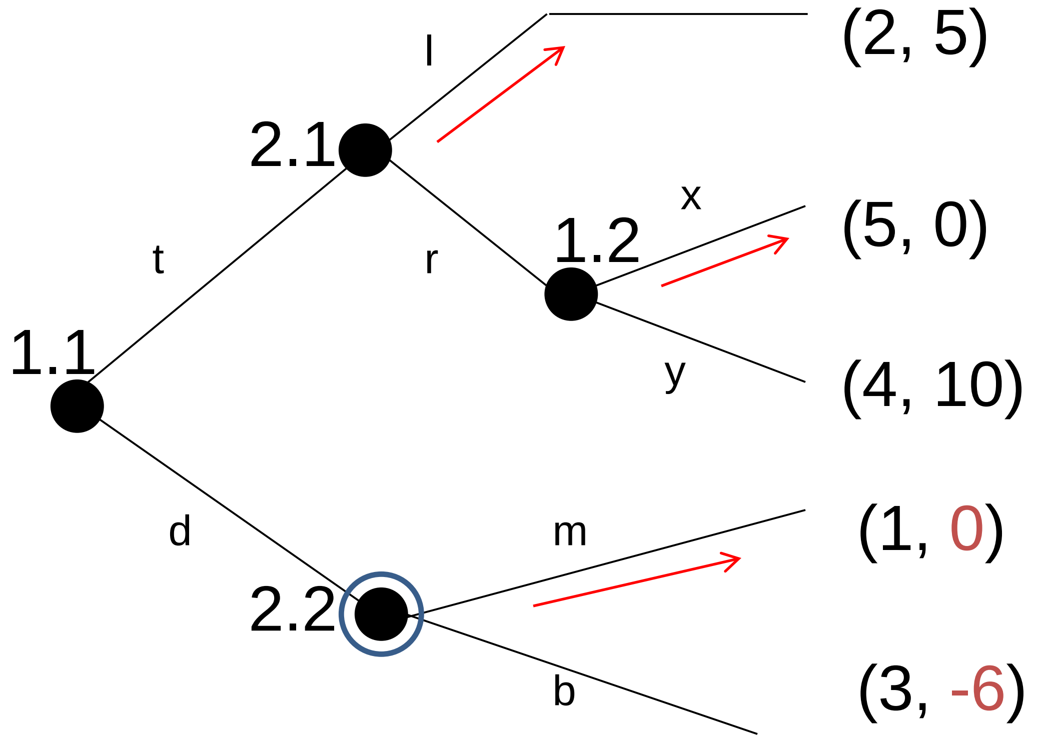
Subgame Perfect Equilibrium



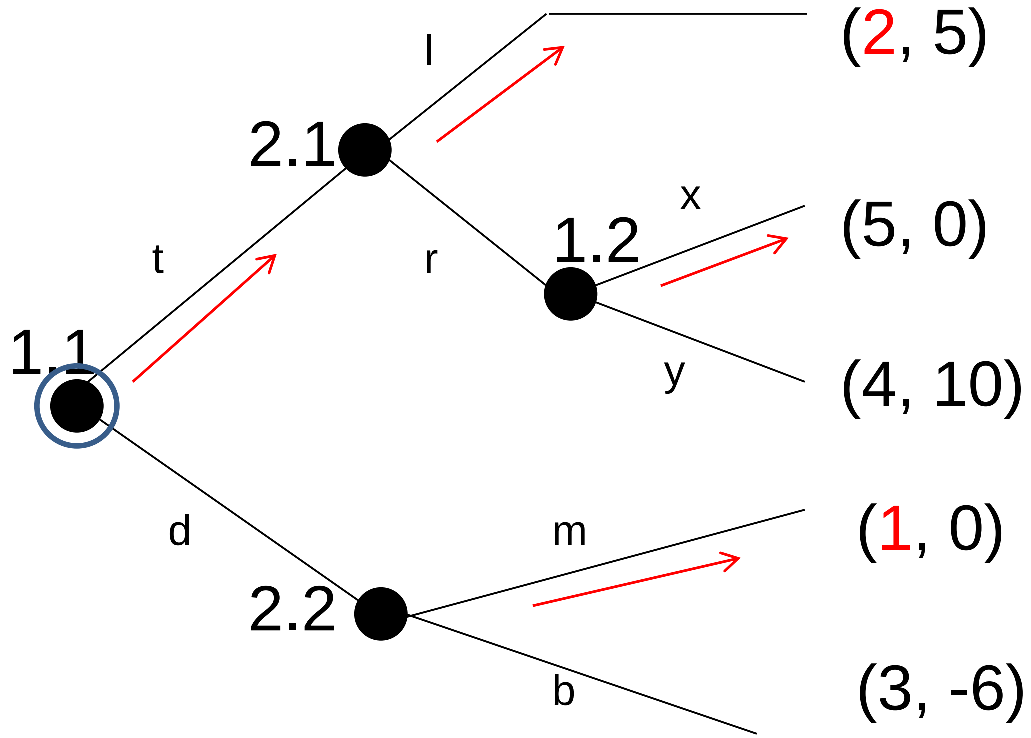
Subgame Perfect Equilibrium



Subgame Perfect Equilibrium

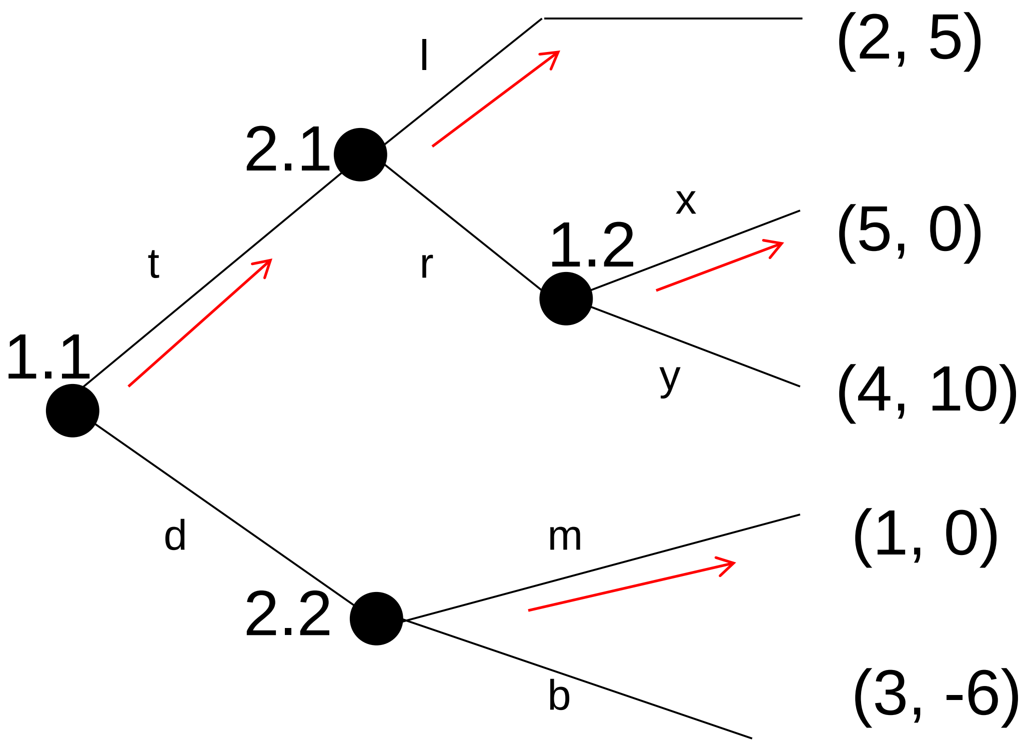


Subgame Perfect Equilibrium



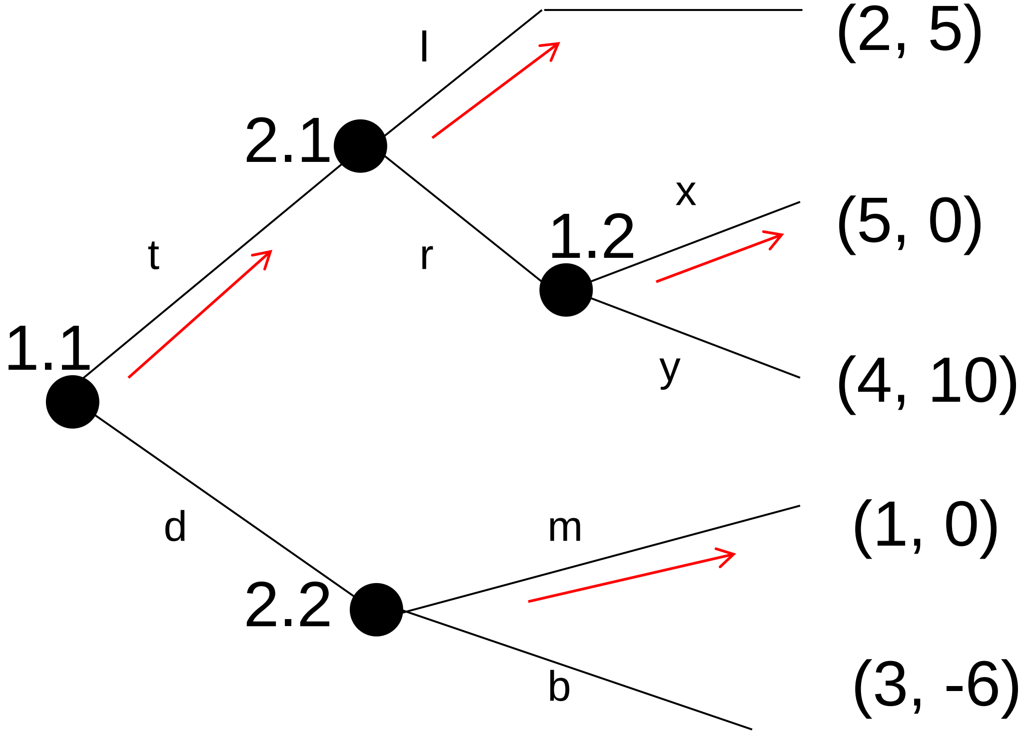
Subgame Perfect Equilibrium

SPE = ?



Subgame Perfect Equilibrium

SPE = {(t,x);(l,m)}



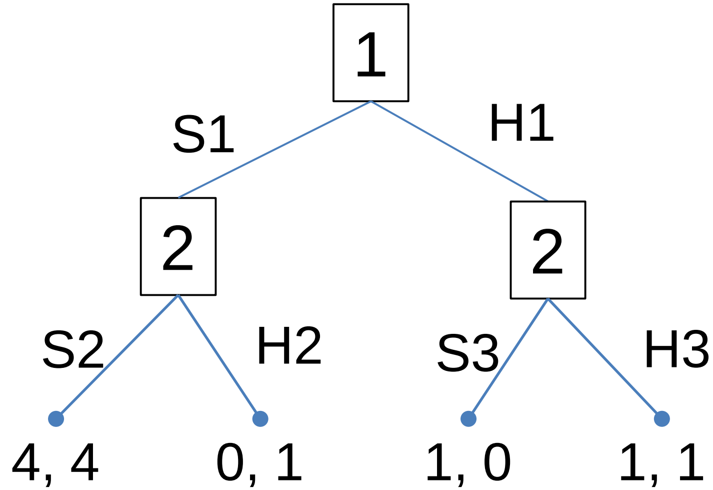
Notice: **If** player 1, was *not* going to play x at 1.2 (he plays y instead), player 2 would play r at 2.1, and player 1 would play t at 1.1. That's why we have to write down player 1's commitment to x in the equilibrium (that commitment binds the equilibrium).

Subgame Perfect Equilibrium

Three Observations:

- Some Nash equilibria are unrealistic in sequential play.
- Rational play in a sequential game requires anticipation. Backward induction captures that anticipation.
- Actions that are not part of the terminal history are essential for SPE because those rational commitments are part of what guarantee the equilibrium.

Example: Sequential stag hunt



What’s the SPE?

Why did we get a different outcome than we did in strategic form?

Extra Credit Game 4 (HW3)

The Pirate Captain's Dilemma

The Pirate Captain’s Dilemma

Description of the Game:

- Seven pirates have just found a treasure chest with 10 gold pieces, which can only be distributed as whole pieces.
- The game proceeds as follows:
  - The current captain makes a proposal about the division of the spoils.
  - The pirates vote. If at least 2/3rds of the pirates agree, the game ends with the agreed upon allocation. If less than 2/3rds of the pirates agree, the captain is thrown to the sharks and the next longest serving pirate becomes captain and makes a proposal.
  - The process repeats until a proposal is accepted.
- Pirates value living first, maximizing the number of gold coins second, and third killing the other pirate if the results are otherwise equal.

What proposal will the initial captain make?

The Pirate Captain's Dilemma

Proposing Pirate	A	B	C	D	E	F	G
Number of Pirates	**7**	**6**	**5**	**4**	**3**	**2**	**1**
Required yeas	**5**	**4**	**4**	**3**	**2**	**2**	**1**
G							10
F						☠	-
E					10	0	0
D				8	0	1	1
C			5	0	1	2	2
B		4	0	1	2	3 ↔	0
A	3	0	1	2	3	0 ↔	1

Median as Agenda Setter

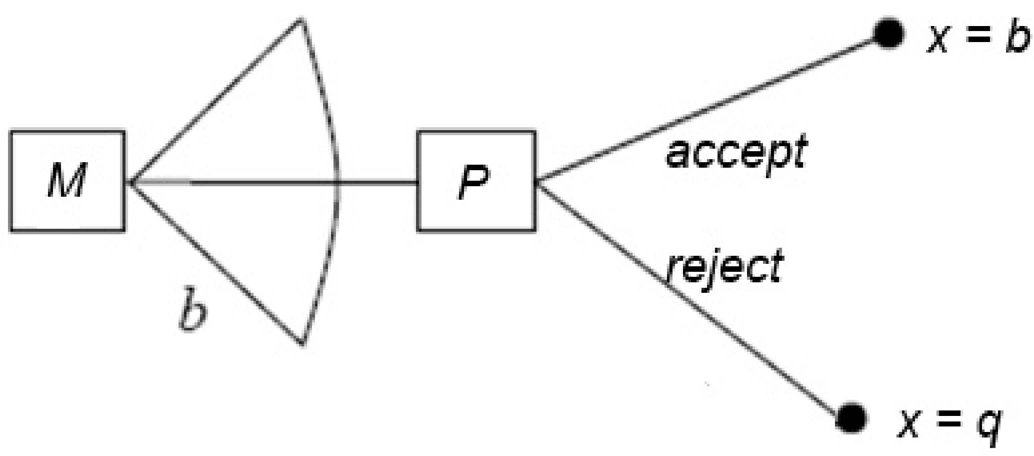
Assume: one chamber, fixed agenda setter, no 2/3rds override.

Median voter (M) proposes a bill b.

President (P) signs bill or vetoes it.

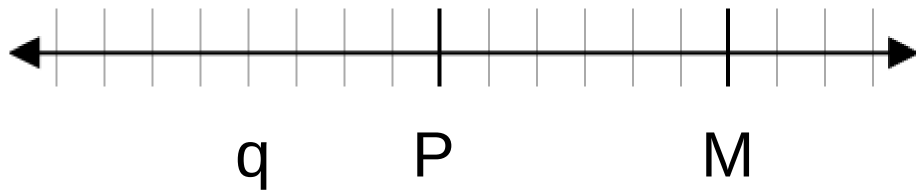
If the president signs, the policy outcome is x = b.

If the president vetoes, the policy outcome is x = q.



Analysis

P = 4, M = 10, q = 0



a. What would M propose?

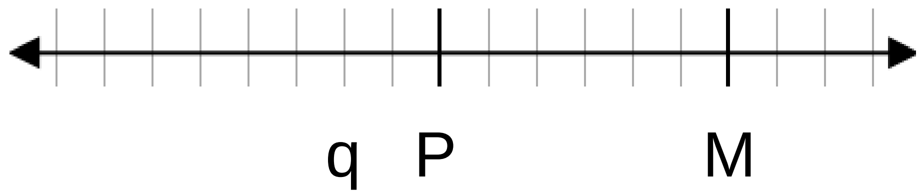
8 – ε, where ε is arbitrarily small.

From here forward, we will just say 8.

b. SPE = {b = 8; accept}

Analysis

P = 4, M = 10, q = 2



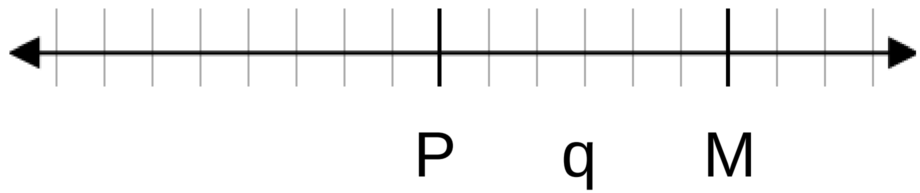
a. What would M propose?

6.

b. SPE = {b = 6; accept}

Analysis

P = 4, M = 10, q = 7



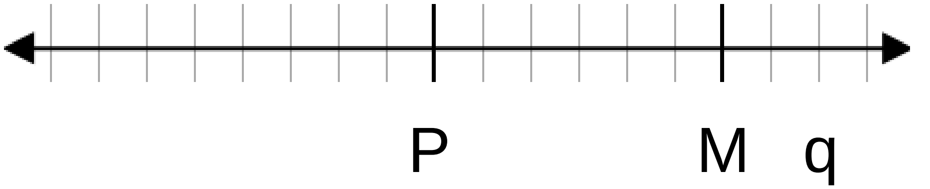
a. What would M propose?

$b \geq 7$

b. SPE = {b ≥ 7; reject}. Outcome: x = 7.

Analysis

P = 4, M = 10, q = 12

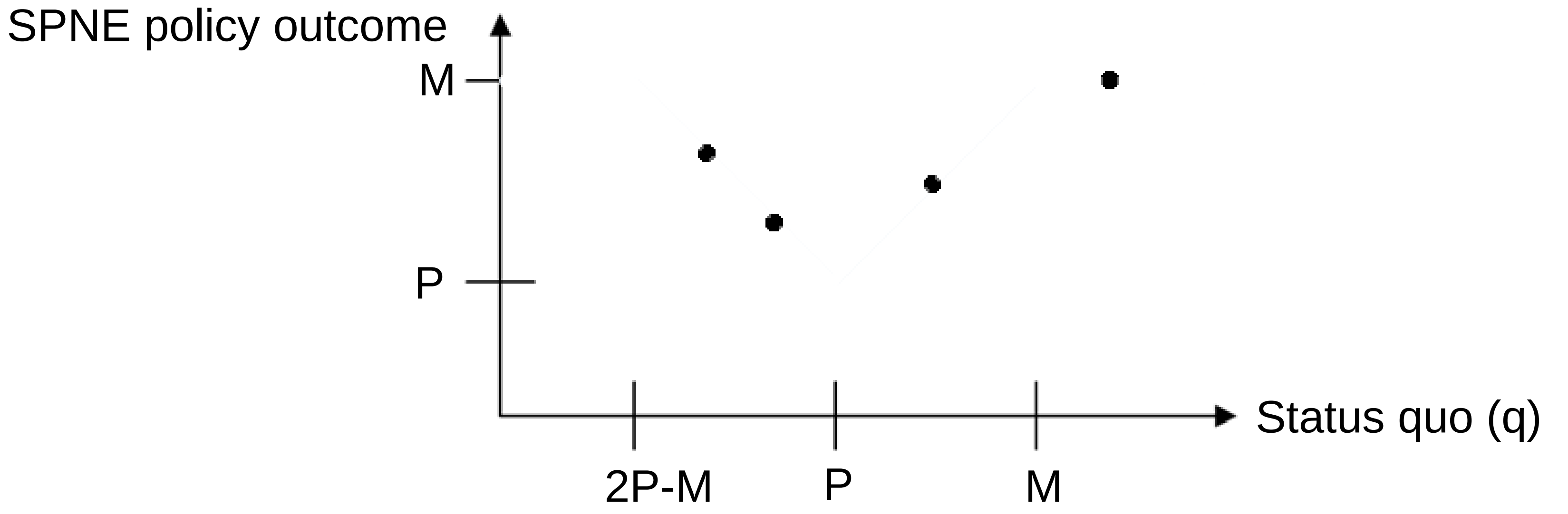


a. What would M propose?

b = A = 10

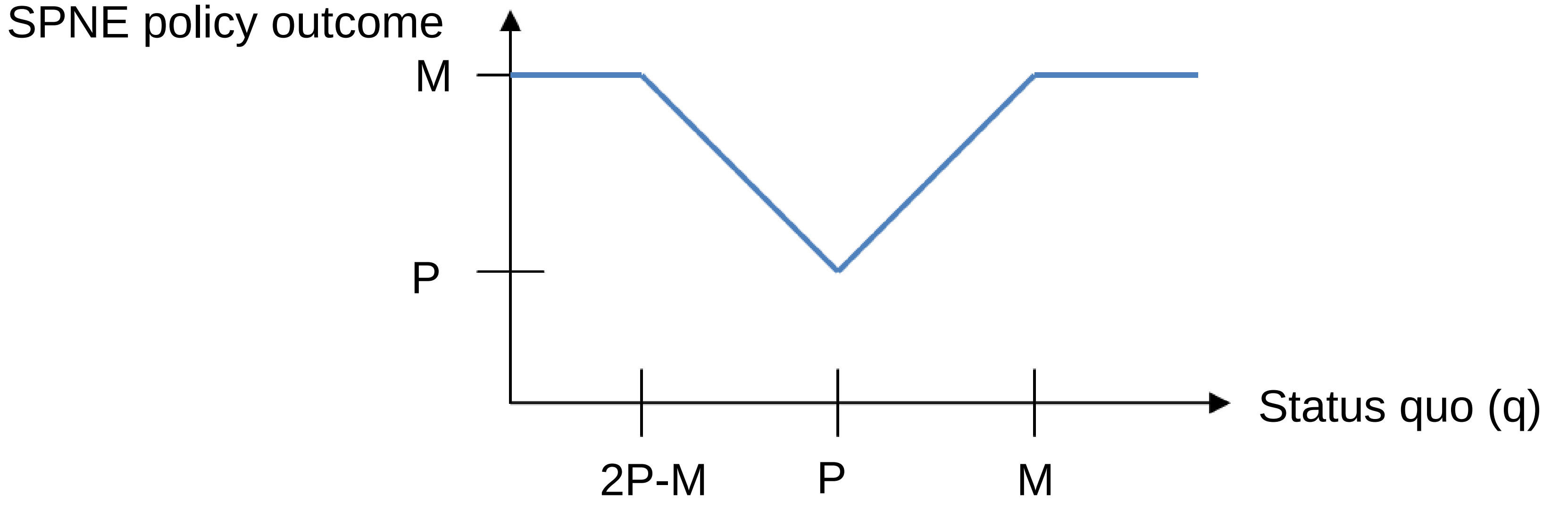
b. SPE = {b = 10; accept}

Comparative statics for q



For the four examples we just did, I mark the outcome on the y-axis given the initial status quo on the x-axis.

Comparative statics for q



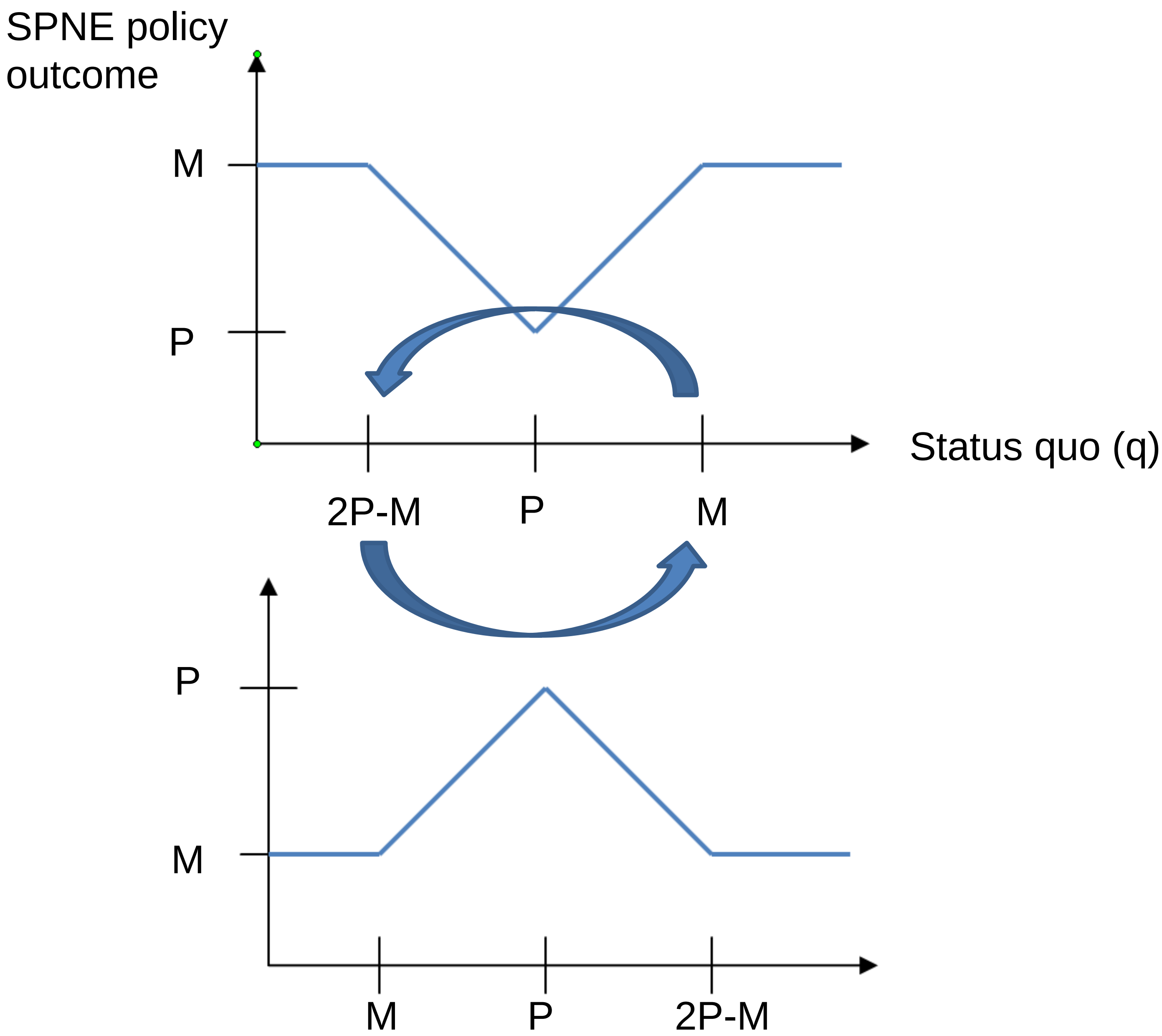
Case I: q < 2P - M

Case III: P < q < M

Case II: 2P – M < q < P

Case IV: M < q

What happens if we switched 2P-M and M?



Implications

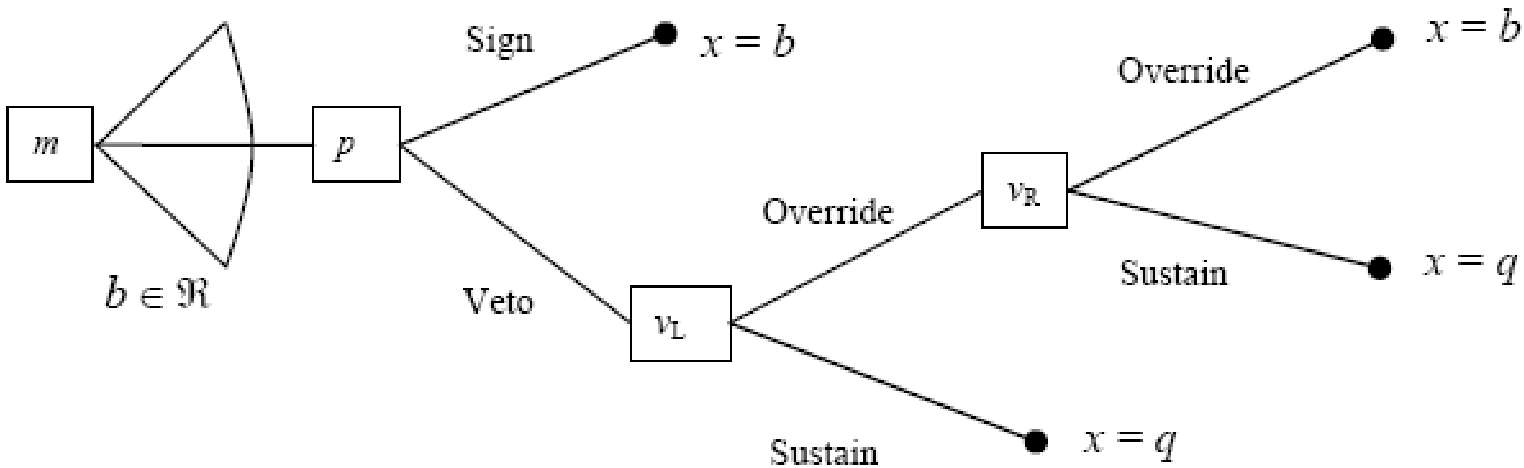
- Provides basic theoretical insight about the roles of *proposal* power and *veto* power.
  - Veto power ensures that outcomes are no worse than the status quo for the president.
- Comparative statics for ideal points
  - Greater distance between M and P $\Rightarrow$ Greater constraint/gridlock.
- Applications
  - Nominating members of Supreme Court: President proposes, Senate may veto.
  - Committees and closed rules: Committee proposes, Chamber must approve of final passage.
  - Judge writes opinion for majority of justices to approve.

One Chamber, Veto override

Assume: median of chamber proposes, president accepts or rejects, veto override.

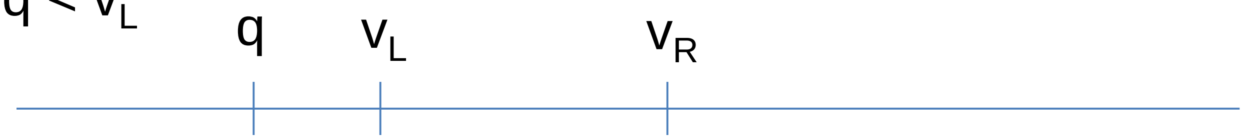
Game Sequence:

1. Median of chamber (M) proposes bill b.
2. President (P) may veto or sign.
3. Congress can override veto with 2/3 majority



Analysis of overrides

$q < v_L$



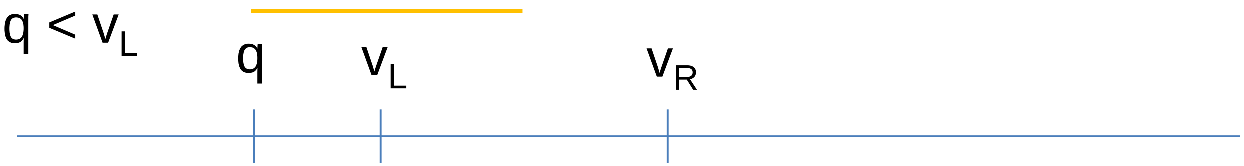
<u>Warm Up</u>:

- v_L is the left 2/3rd pivot
- v_R is the right 2/3rd pivot.

- Question: what points could attain a 2/3rds override of q?

Analysis of overrides

$q < v_L$



$\color{orange}{W_{V_L}(Q)} = W_{V_L}(Q) \cap W_{V_R}(Q)$

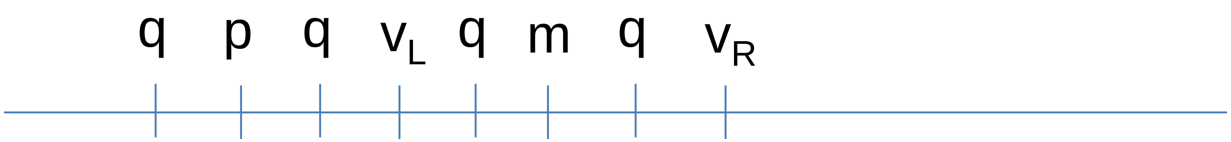
$v_L < q < v_R$

$W_{V_L}(Q) \cap W_{V_R}(Q) = \emptyset$

$v_R < q$

$\color{orange}{W_{V_R}(Q)} = W_{V_L}(Q) \cap W_{V_R}(Q)$

SE for various positions of q

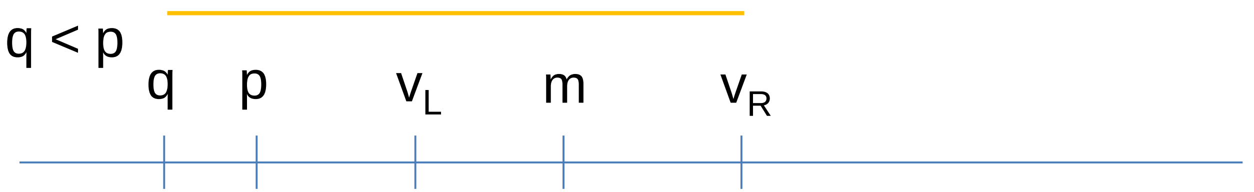


- Assume: $p < v_L < m$.

- We will examine four possible locations of q:
  - $q < p$
  - $p < q < v_L$
  - $v_L < q < m$
  - $m < q$

- A more complete analysis would also include:
  $m < v_R < p$
  $m < p < v_R$
  $v_L < p < m$.

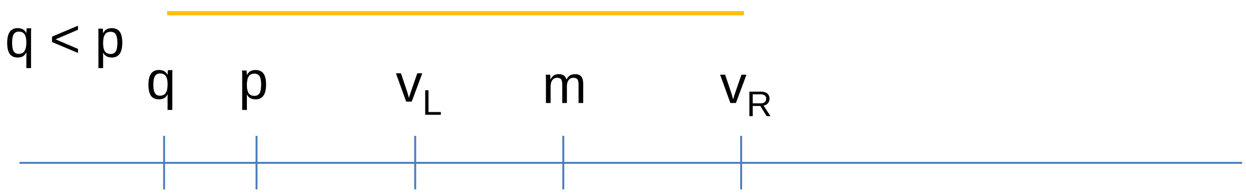
Analysis of vetoes and proposals



Solve by backward induction:

First, graph what could attain 2/3rds override.

Analysis of vetoes and proposals



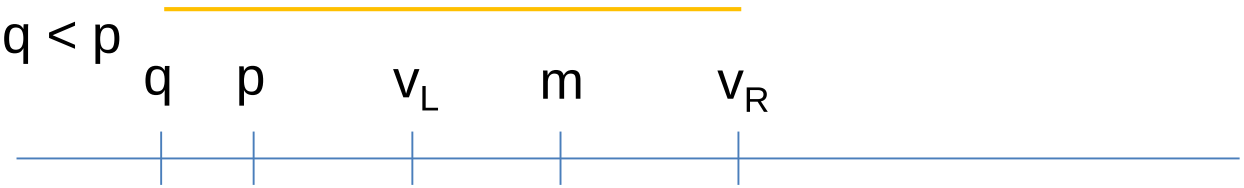
President could sign or veto, because she cannot affect the outcome (if m proposes rationally, it will pass).

Second, decide whether the president signs or vetoes.

Because of technicalities like this, sometimes it is easier to skip the President and come back to her later.

Analysis of vetoes and proposals

q < p



Third, consider what m would propose.

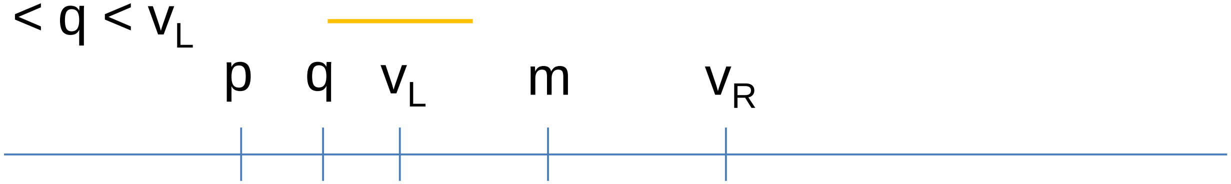
m will propose m because m is in $W_{V_L}(Q)$ which will pass.

Hence, m is the outcome.

SPE = {b=m; vetoes; override}
{b=m; accepts; override}
{b=m; accepts; sustain}

Analysis of vetoes and proposals

$p < q < v_L$

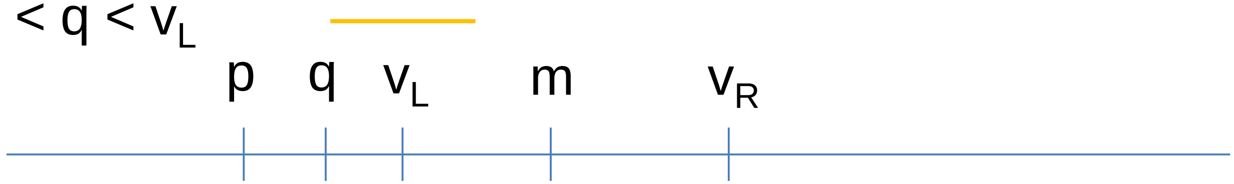


Solve by backward induction:

First, graph overrides.

Analysis of vetoes and proposals

$p < q < v_L$

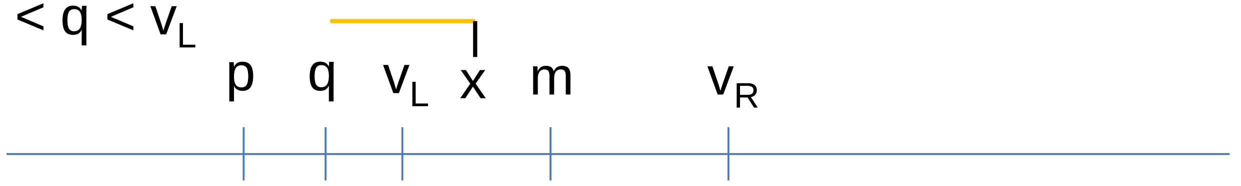


Second, decide whether president signs or vetoes.

President could sign or veto, because she cannot affect the outcome (if m proposes rationally, it will pass).

Analysis of vetoes and proposals

$p < q < v_L$



Third, consider what m would propose.

m will propose x because x is the element closest to m that is in $W_{V_L}(Q)$.

Hence, x is the outcome.

SPE = {b=x; vetoes; override}
{b=x; accepts; override}
{b=x; accepts; sustain}

Analysis of vetoes and proposals

$v_L < q < m$

First, graph overrides.

$W_{v_L}(Q) \cap W_{v_R}(Q) = \emptyset$

Analysis of vetoes and proposals

$v_L < q < m$

President vetoes because m wants to move the bill to the right.

Second, decide whether the president signs or vetoes.

Analysis of vetoes and proposals

$v_L < q < m$

m cannot propose anything that passes, so m proposes a throw away (i.e. any x: x > q).

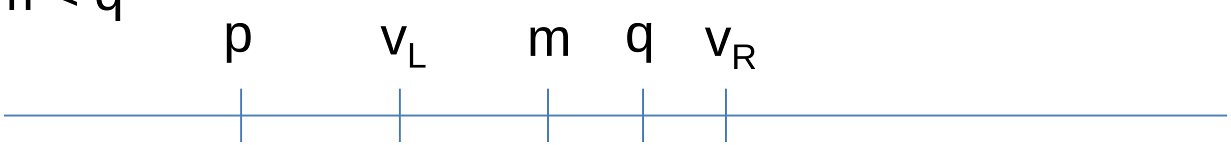
Outcome: q

SPE = {b=x > q; vetoes; sustain}

Third, consider what m would propose.

Analysis of vetoes and proposals

$m < q$



First, graph overrides.

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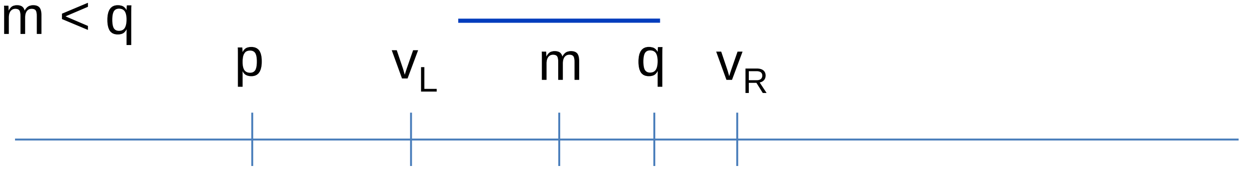
Analysis of vetoes and proposals

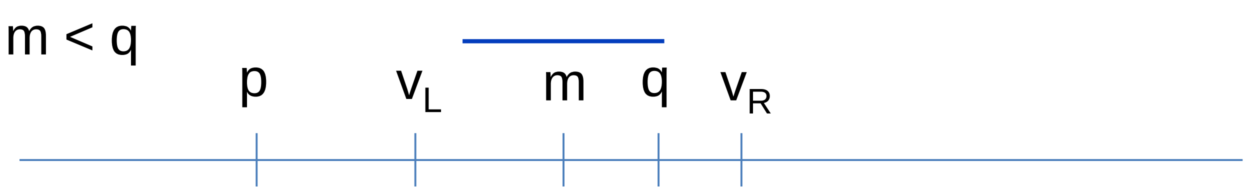
$m < q$



Second, decide whether the president signs or vetoes.

President signs anything in $W_m(Q)$ because he prefers that to q.

Analysis of vetoes and proposals

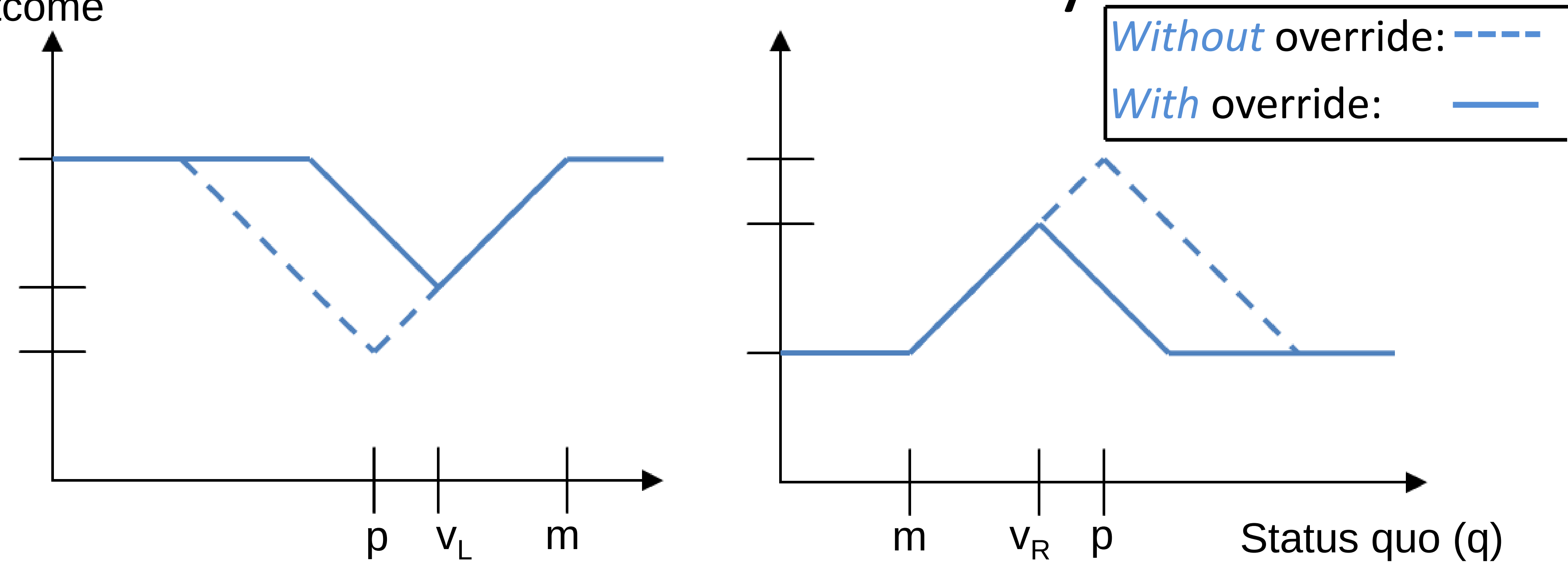


Third, consider what m would propose.

m will propose m because m is in $W_m(Q)$ which will pass.

Outcome: m.

Veto model summary



- Although there are two veto pivots, only the veto pivot *closest to the president's ideal* point is relevant.
- If the president is farther from m than the relevant veto pivot, then the median legislator's proposal is *constrained by the veto pivot's preferences* rather than the president's.