THEORY: NASH EQUILIBRIUM

Prisoner's Dilemma

The Story

- Two prisoners held in separate rooms.
- Authorities offer a reduced sentence to each prisoner if he rats out his friend.
- If a prisoner is ratted out by the other guy, then he receives a harsh sentence. If he rats out the other guy, then he receives a lighter sentence.

Matrix representation

		Player 2 (Column)	
		C	**D**
Player 1 (Row)	**C**	2 , 2	0 , 5
	D	5 , 0	1 , 1

Numbers are utility, not years. Hence, larger numbers are more preferred.

I need a volunteer

Model of a strategic game

A **strategic (or normal form) game** consists of:

- A set of **players,** denoted by $N = \{1,\ldots,n\}$,
- For every player $i \in N$, a set of **strategies** $S_i = \{s_1, s_2, \ldots, s_k\}$,
- The set of **strategy profiles** $S \equiv \prod S_i$, which are the possible outcomes of the game,
- For every player, preferences over the set of strategy profiles:

$$u_i : S \to \mathbb{R}.$$

Matrix representation

Set of Players:
{player 1, player 2}.

Player 2
(Column)

Player 1
(Row)

	C	D
C	2 , 2	0 , 5
D	5 , 0	1 , 1

Set of Strategies for
player 1: {C, D}.

Player 1’s preferences for the
strategy profiles (in utility).

Set of Strategies Profiles:
{(C,C),(C,D),(D,C),(D,D)}

Model of a strategic game

More on strategy profiles:

- For any strategy profile $s \in S$, we will use the notation s_{-i} to denote the strategies adopted by all players **except** player *i*, that is $s = (s_i, s_{-i})$.

Utility Functions:

- A utility function for player *i* is a function $u_i : S \to \mathbb{R}$ the represents the player's preferences over the strategy profiles.

Common Knowledge

Definition:

- A fact is **common knowledge** if all of the players know it, and all of the players know that all of the players know it, etc.

Rationality:

- We assume that it is common knowledge among all players that they are rational.
  - That is, they want to get the outcomes they prefer most among the ones they can actually attain.

Remarks

Time

- Time is absent from strategic form games. Players cannot make their actions contingent on the actions of other players, perhaps because
  - Players act simultaneously,
  - Players are not informed about the previous moves of the other players.

Information

- We are currently assuming players have complete information. They know the structure of the game, the strategies available, and the preferences of all players.
  - The last assumption will be relaxed when we get to Bayesian Subgame Perfect Equilibrium.

Stag Hunt

Player 1 (Row) \ Player 2 (Column)	S	H
S	3 , 3	0 , 1
H	1 , 0	1 , 1

Extra credit on HW2. I'm going to randomly match you with someone in the room. Imagine you are row. How would you play this game?

Chicken

		Player 2 (Column)	
		S	**H**
Player 1 (Row)	**S**	2 , 2	1 , 3
	H	3 , 1	-1 , -1

How would two rational actors play this game?

Solution Concepts

Definition

- A **solution concept** is a tool for making a prediction about how rational players are going to play a game. It identifies some strategy profile as more plausible than others.

Desirable Properties of Solution Concepts

- Existence: the concept should apply to a wide class of games.
- Exclusivity: the concept should narrow down the list of strategy profiles.
- Robustness: small changes in the game should not affect the prediction made by the solution concept.

Nash equilibrium

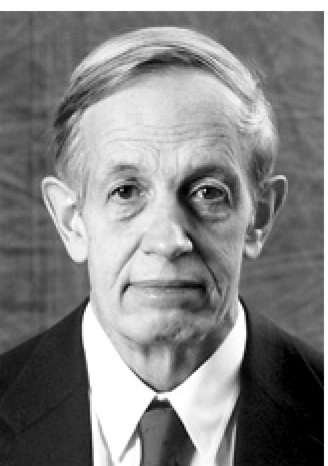
Informal definition

- A Nash equilibrium (NE) is a strategy profile such that no player has a unilateral incentive to "deviate" (if the strategies of all the other players are held constant, no player would like to change his/her strategy).

Formal definition

- A strategy profile $s^* \in S$ is a **Nash Equilibrium** in a strategic form game G if and only if $\forall i \in N$ and $\forall s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$



Example: PD

		Column	
		C	**D**
Row	**C**	3, 3	0, 5
	D	5, 0	1, 1

{D;D} is a Nash Equilibrium in this game, because neither player has a unilateral incentive to deviate.

Remarks

- Indifference keeps a player in equilibrium. In order to have an "incentive to deviate" a player's utility from another action must be *strictly* better than s*.
- A Nash equilibrium is a *"stable" outcome* in the sense that it is self-enforcing.
- There may be multiple Nash equilibria in a game.
- Nash equilibria are not necessarily efficient. All players may unanimously prefer another outcome to a Nash Equilibrium.

Best Response Function

Definition

- The **best response function** for player *i* is defined by

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(\tilde{s}_i, s_{-i}) \; \forall \tilde{s}_i \in S_i\}.$$

Theorem

- A strategy profile s* is a NE if and only if $\forall i \in N, s_i^* \in B_i(s_{-i}^*)$

Implication

- We can use the best response function to identify Nash Equilbria. Nash Equilibria occur where the best response functions for all the players intersect.

Nash Equilibrium

Column

Row		C	D
	C	3, 3	0, 2
	D	2, 0	1, 1

How do I find Nash equilibria?

Determine the best responses, that is the best strategy for a player given the strategies played by opponents. The best responses for each player intersect at the Nash equilibrium.

Nash Equilibrium

Column

Row		C	D
	C	3, 3	0, 2
	D	2, 0	1, 1

Given column plays C, what is best response for Row?

Nash Equilibrium

Column

Row

	C	D
C	3, 3	0, 2
D	2, 0	1, 1

Given column plays C, what is best response for Row?

C because 3 > 2.

Nash Equilibrium

Column

Row	C	D
C	3, 3	0, 2
D	2, 0	1, 1

Given column plays C, what is best response for Row?

C because 3 > 2.

Let’s circle 3 because it indicates one of the best responses.

Nash Equilibrium

Column ↓

Row

	C	D
C	3, 3	0, 2
D	2, 0	1, 1

Given column plays D, what is best response for Row?

Nash Equilibrium

Column ↓

Row

	C	D
C	③, 3	0, 2
D	2, 0	1, 1

Given column plays D, what is best response for Row?

D because 1 > 0.

Nash Equilibrium

Column ↓

Row		C	D
	C	(3), 3	0, 2
	D	2, 0	(1), 1

Given column plays D, what is best response for Row?

D because 1 > 0.

Let’s circle 1 because it indicates one of the best responses.

Nash Equilibrium

Column

Row →

	C	D
C	(3), 3	0, 2
D	2, 0	(1), 1

Given "Row" plays C, what is best response for Column?

Nash Equilibrium

Column

	C	D
C	3, 3	0, 2
D	2, 0	1, 1

Row

Given “Row” plays C, what is best response for Column?

C because 3 > 2.

Nash Equilibrium

Column

Row		C	D
	C	3, 3	0, 2
	D	2, 0	1, 1

Given “Row” plays C, what is best response for Column?

C because 3 > 2.

Let’s circle 3 because it indicates one of the best responses.

Nash Equilibrium

Column

Row

	C	D
C	3, 3	0, 2
D	2, 0	1, 1

Given “Row” plays D, what is best response for Column?

Nash Equilibrium

Column

Row

	C	D
C	3, 3	0, 2
D	2, 0	1, 1

Given “Row” plays D, what is best response for Column?

D because 1 > 0.

Nash Equilibrium

Column

Row →

	C	D
C	③,③	0, 2
D	2, 0	①,①

Given "Row" plays D, what is best response for Column?

D because 1 > 0.

Let's circle 1 because it indicates one of the best responses.

Nash Equilibrium

Column

Row	C	D
C	3, 3	0, 2
D	2, 0	1, 1

Where the best responses intersect {C;C} and {D;D} are Nash Equilbria.

N.E. = {C;C} and {D;D}

Note: equilibria are always stated in terms of strategies, never in terms of payoffs.

Practice: Chicken

		Player 2 (Column)	
		S	**H**
Player 1 (Row)	**S**	2 , 2	1 , 3
	H	3 , 1	-1 , -1

What's the Nash Equilibrium in this game?

Hint: use best responses

Practice: Nash Equilibrium

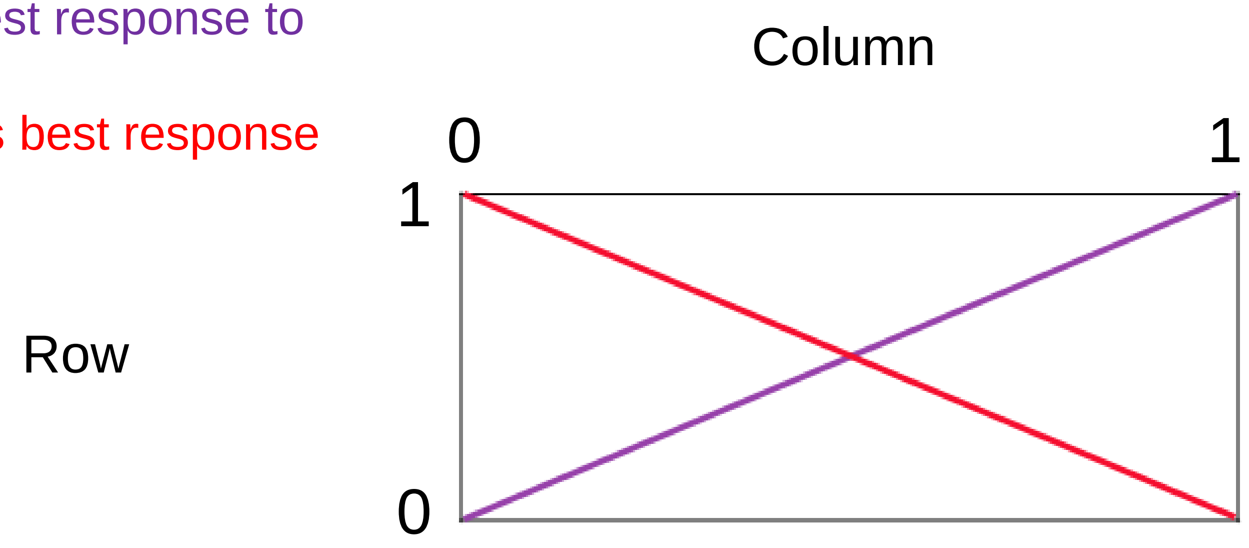
		Column x	Column y	Column z
Row	A	2, 3	-16, 2	5, 0
Row	B	5, 6	4, 6	6, 4
Row	C	8, 0	3, 10	1, 8

Practice 2: Nash Equilibrium

		Column		
		None	Some	All
Row	All	0, 20	5, 15	10, 10
	Some	1, 3	4, 4	15, 5
	None	2, 2	3, 1	20, 0

Continuous Strategies

Row's best response to Column.
Column's best response to Row.



Rather than making a discrete choice to contribute a none, some, or all, we could consider making a continuous choice between 0 (none), 1 (all), and everything in between.

Continuous strategies are common in spatial voting models.

Exercise 42.2 (A Joint Project)

Description of the Game

- Two people are working on a joint project (like a group homework assignment). Each person must chose an effort level $x \in [0, 1]$. Effort costs $c(x_i) = x_i^2$. The benefit of their efforts is $f(x_1, x_2) = 3x_1x_2$, which is split equally among them.

Utility Functions

- The utility for each individual is:

$$u_i(x_i, x_j) = \frac{3}{2}x_ix_j - x_i^2$$

Exercise 42.2 (A Joint Project)

$$u_i(x_i, x_j) = \frac{3}{2}x_i x_j - x_i^2$$

Find Best Response Functions:

First, maximize player 1's utility with respect to her effort:

$$\frac{\partial u_i(x_i, x_j)}{\partial x_1} = \frac{3}{2}x_2 - 2x_1$$

Solving the first order condition (FOC) for x_1 (i.e., setting the derivate equal to 0 and solving for x_1) yields the best response for player 1 to player 2's choice effort:

$$x_1^* = B_1(x_2) = \frac{3}{4}x_2.$$

Symmetrically, the best response function for player 2 is:

$$x_2^* = B_2(x_1) = \frac{3}{4}x_1.$$

Exercise 42.2 (A Joint Project)

Nash Equilibrium:

In equilibrium, both players must be playing a best response to the other player's effort level. Mathematically, it must be that:

$$x_1^* = \frac{3}{4}x_2^*$$

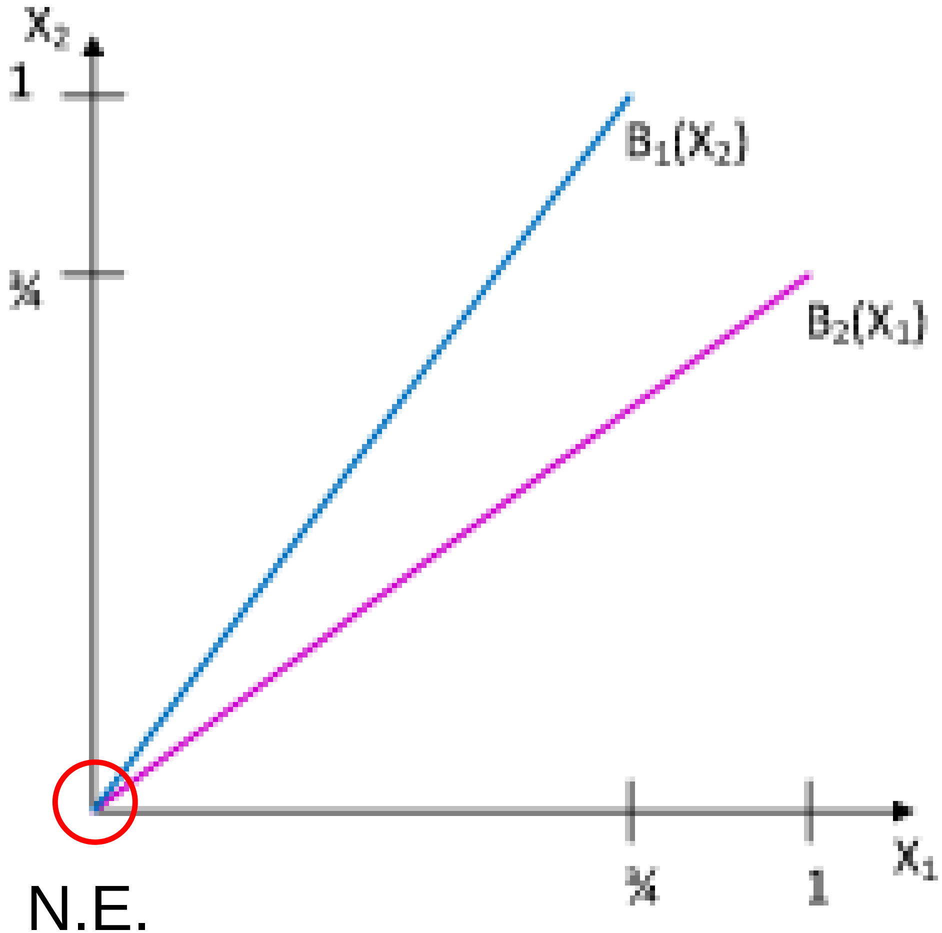
$$x_2^* = \frac{3}{4}x_1^*$$

Substituting yields:

$$x_2^* = \frac{9}{16}x_2^* \leftrightarrow x_2^* = 0.$$

This implies $x_1^* = 0.$

Exercise 42.2 (A Joint Project)



Application: Electoral competition

Baseline model: Office-motivated candidates

- Hotelling (1929), Downs (1957)
- Two candidates choose policy platforms
- Platforms credibly translate into policy outcomes
- Candidates care only about winning
  - they prefer wining to tying to losing
- Voters
  - Care about policy outcomes
  - Single-peaked and symmetric preferences
  - Continuous distribution of voters with median m
  - Voters are non-strategic

Baseline model: Office Motivated Candidates

Players Candidates, $N = \{1, 2\}$

Strategies Platforms, $X_i = \mathbb{R}$.

Preferences

$$u_1(x_1, x_2) = \begin{cases} 1 & \text{if } |x_1 - m| < |x_2 - m| \\ 0 & \text{if } |x_1 - m| = |x_2 - m| \\ -1 & \text{if } |x_1 - m| > |x_2 - m| \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} 1 & \text{if } |x_1 - m| > |x_2 - m| \\ 0 & \text{if } |x_1 - m| = |x_2 - m| \\ -1 & \text{if } |x_1 - m| < |x_2 - m| \end{cases}$$

Note: continuous strategies, but not continuous payoffs. Hence, we won't use first derivatives.

Nash equilibrium

Proposition (m, m) is the unique Nash equilibrium

Proof

Step 1. Show that (m, m) is a NE

Nash equilibrium

Step 2. Show that no other (x_1, x_2) is a NE

Best response functions for player 1

Case1. $x_2 < m$

Case 2. $x_2 = m$

Case 3. $x_2 > m$

Summary

- Strategic games
  - Players
  - Strategies for each player
  - Preferences over strategy profiles

- Nash equilibrium
  - Strategy profile such that no player has a unilateral incentive to deviate
  - Best Response Functions (for both discrete and continuous strategies)
  - Predicts stable outcomes, but may not be unique