IMPERFECT INFORMATION
(SUB-GAME PERFECT BAYESIAN EQUILIBRIUM)

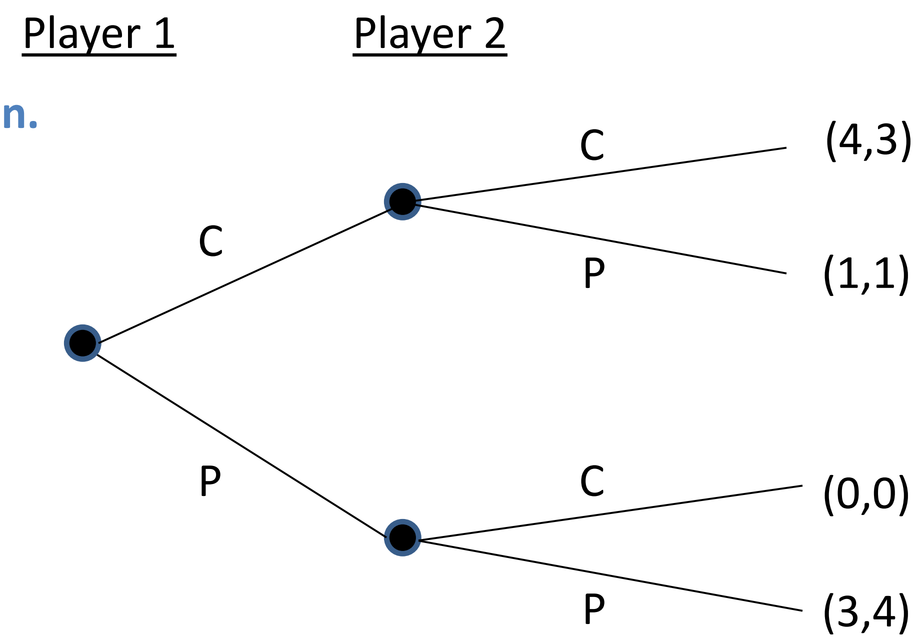
Perfect vs imperfect information

- **Perfect** information
  - When making a move, a player has perfectly observed all previously actions chosen.
    - For each decision, they know exactly where they are in the tree.
- **Imperfect** information
  - When making a move, a player may not know all previous actions chosen.
    - For at least one decision, they don't know where they are in the tree.
- **Information sets**
  - A collection of histories that a player cannot distinguish when it is her turn to move.
    - Information sets can contain multiple histories or one history.
    - If multiple histories, then you have imperfect information.

EX: Battle of the Sexes

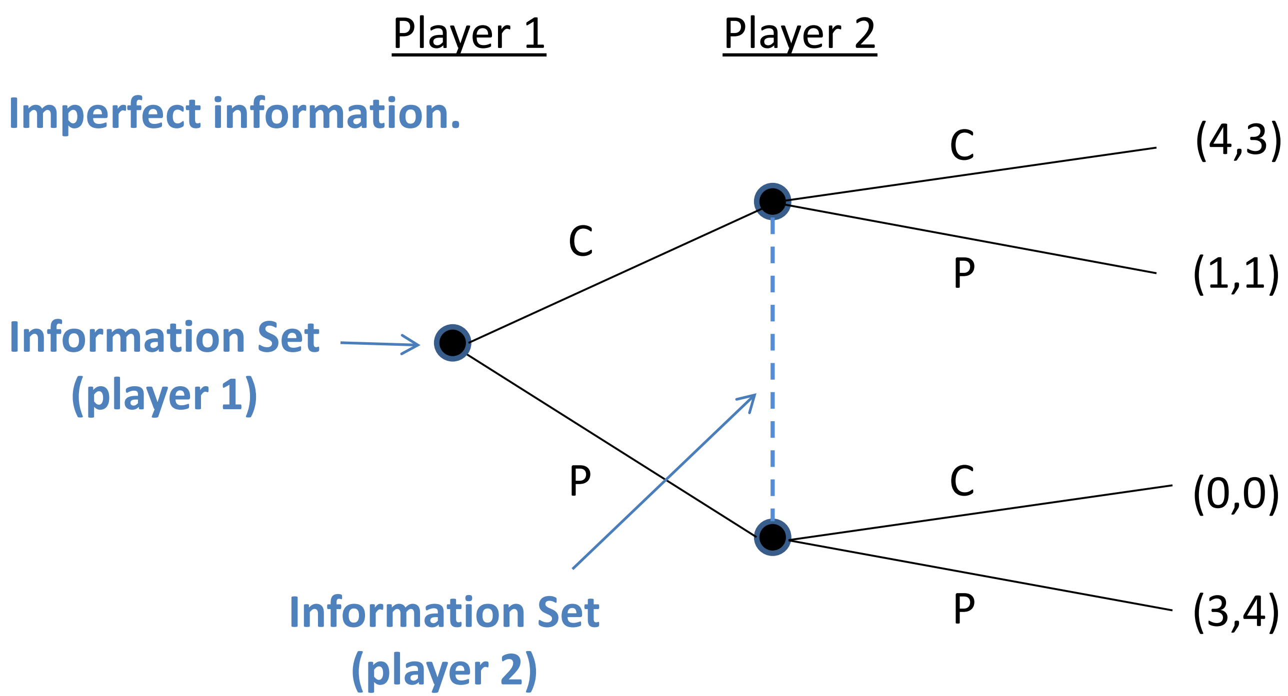
Two PhD students, using a dating service, decide whether to meet in a pub or a café. They just forgot where to meet.

Perfect information.

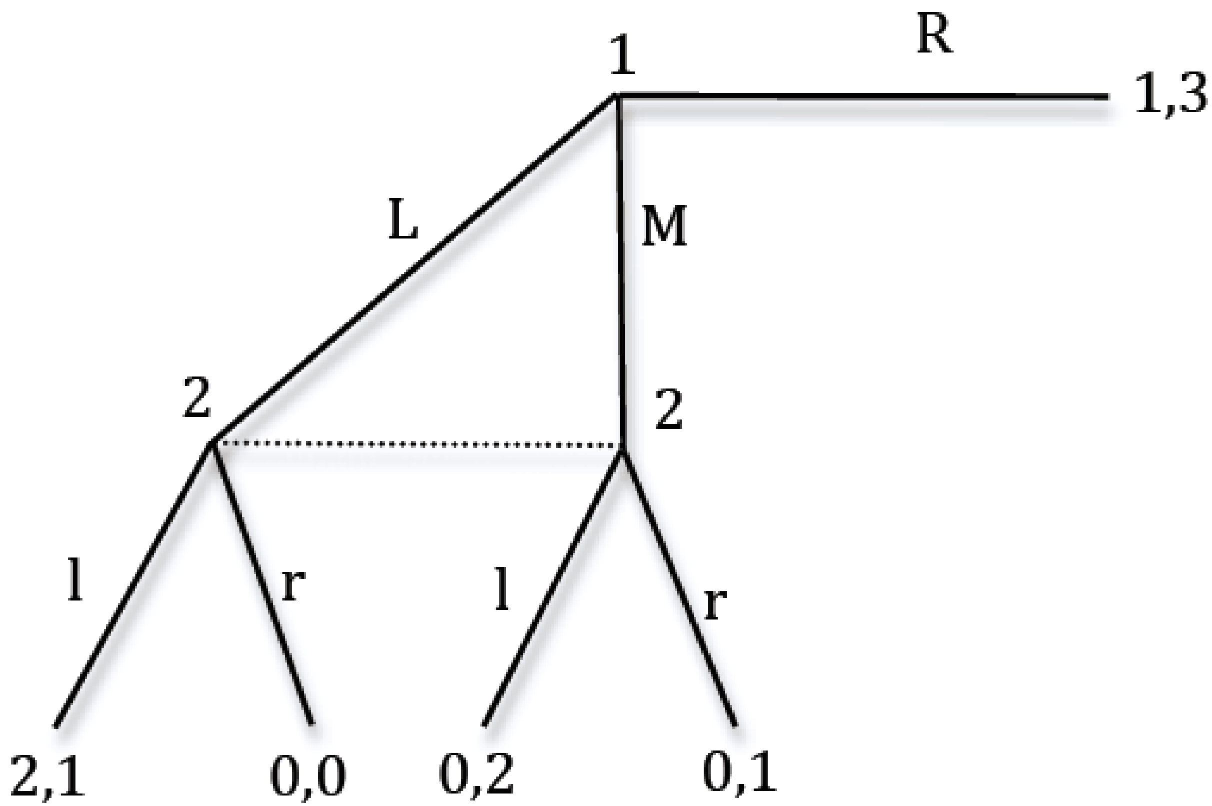


EX: Battle of the Sexes 2

Two PhD students, using a dating service, decide whether to meet in a pub or a café. They just forgot where to meet.



Extensive Form Games with Imperfect Information



Extensive Form Games with Imperfect Information

Strategies

- A strategy in an extensive form game with imperfect information must specify the action that each player will take at each of the player's *information* sets.

Subgames

- Sub-games cannot "cut" information sets (i.e., information sets cannot be divided).
  – Hence, for many extensive form games, the entire game is the only subgame, reducing SPE to simple Nash equilibrium.

Extensive Form Games with Imperfect Information

Recovering Subgame-Perfect equilibrium

- To recover the spirit of the subgame-perfect refinement, we would like to ensure that players act optimally at all of their information sets.

Sequential Equilibrium (S.E.)

- Sequential Equilibrium is our first attempt at doing this. Loosely, a sequential equilibrium is a Nash Equilibrium that maintains consistent beliefs and is sequentially rational.

Extensive Form Games with Imperfect Information

Beliefs

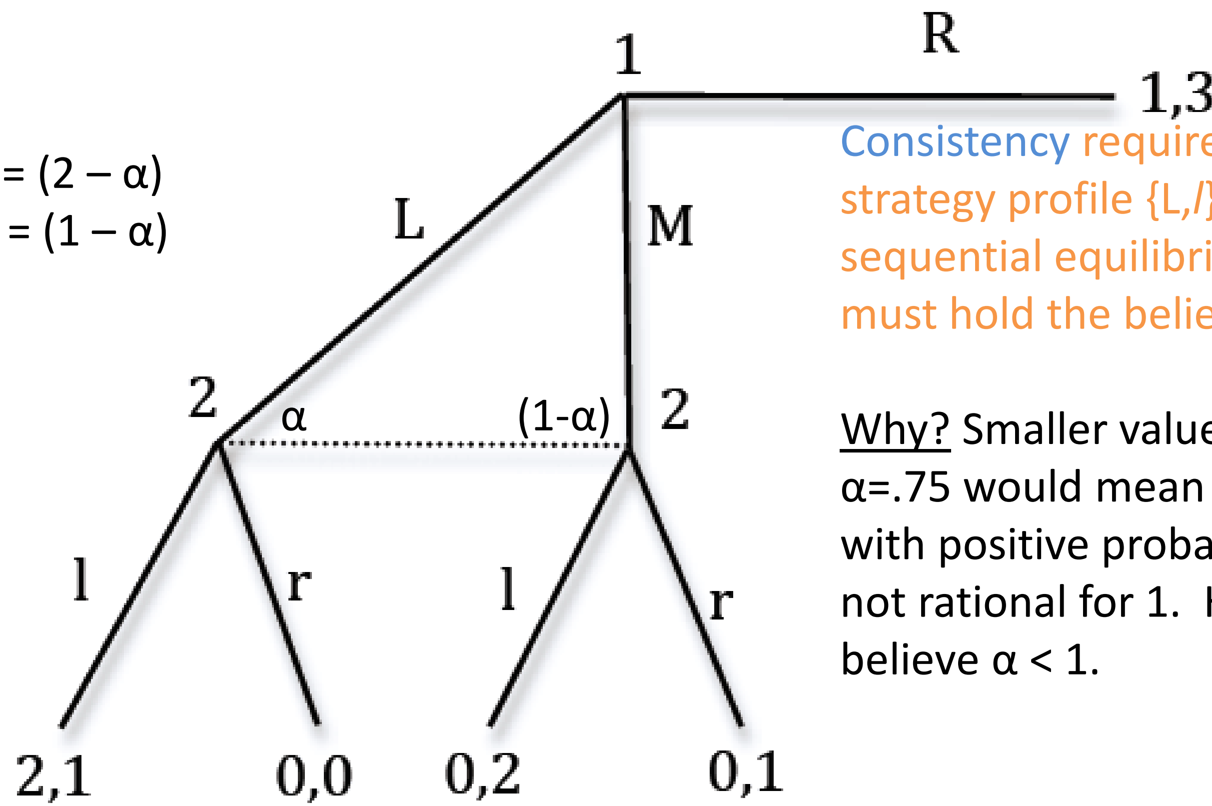
- At every information set, a player must hold a belief about which history has occurred. A belief is represented by a probability distribution over the histories in the information set.

Consistent Beliefs

- Along the path of play, each player's "beliefs" at her information sets must be *consistent* with the strategy profile being played.

Extensive Form Games with Imperfect Information

$EU_2(l) = (2 - \alpha)$
$EU_2(r) = (1 - \alpha)$



Consistency requires that for the strategy profile {L,*l*} to be a sequential equilibrium, player 2 must hold the belief that $\alpha = 1$.

<u>Why?</u> Smaller values of α, like α=.75 would mean M is played with positive probability, which is not rational for 1. Hence, 2 can't believe $\alpha < 1$.

S.E. = {L; (l | α = 1)}

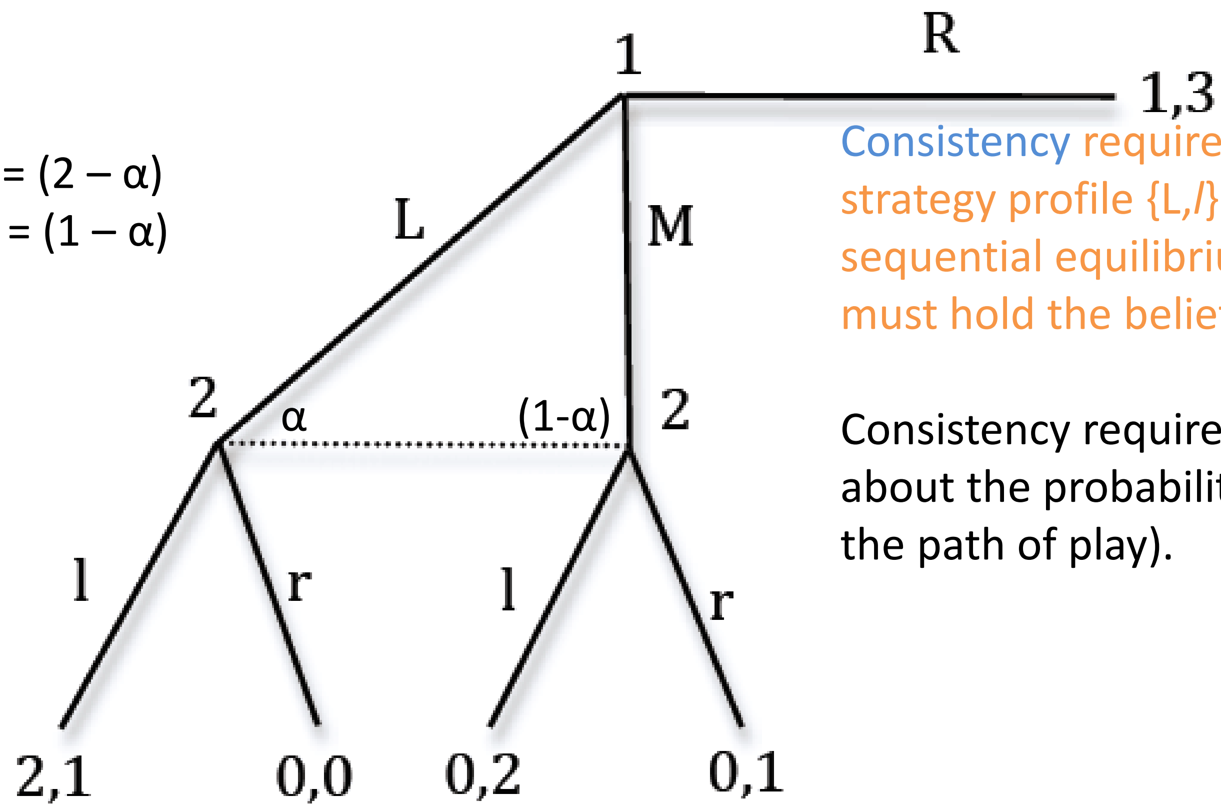
Extensive Form Games with Imperfect Information

Beliefs Off the Path

- At information sets that are off the path of play, i.e. not reached with positive probability, consistency imposes no restrictions on the beliefs a player may hold. Therefore, the player may hold any beliefs.
  - This allows analysts to construct whatever beliefs are useful off the path of play.

Extensive Form Games with Imperfect Information

$EU_2(l) = (2 - \alpha)$
$EU_2(r) = (1 - \alpha)$



Consistency requires that for the strategy profile {L,*l*} to be a sequential equilibrium, player 2 must hold the belief that α = 1.

Consistency requires nothing about the probability of R (it's off the path of play).

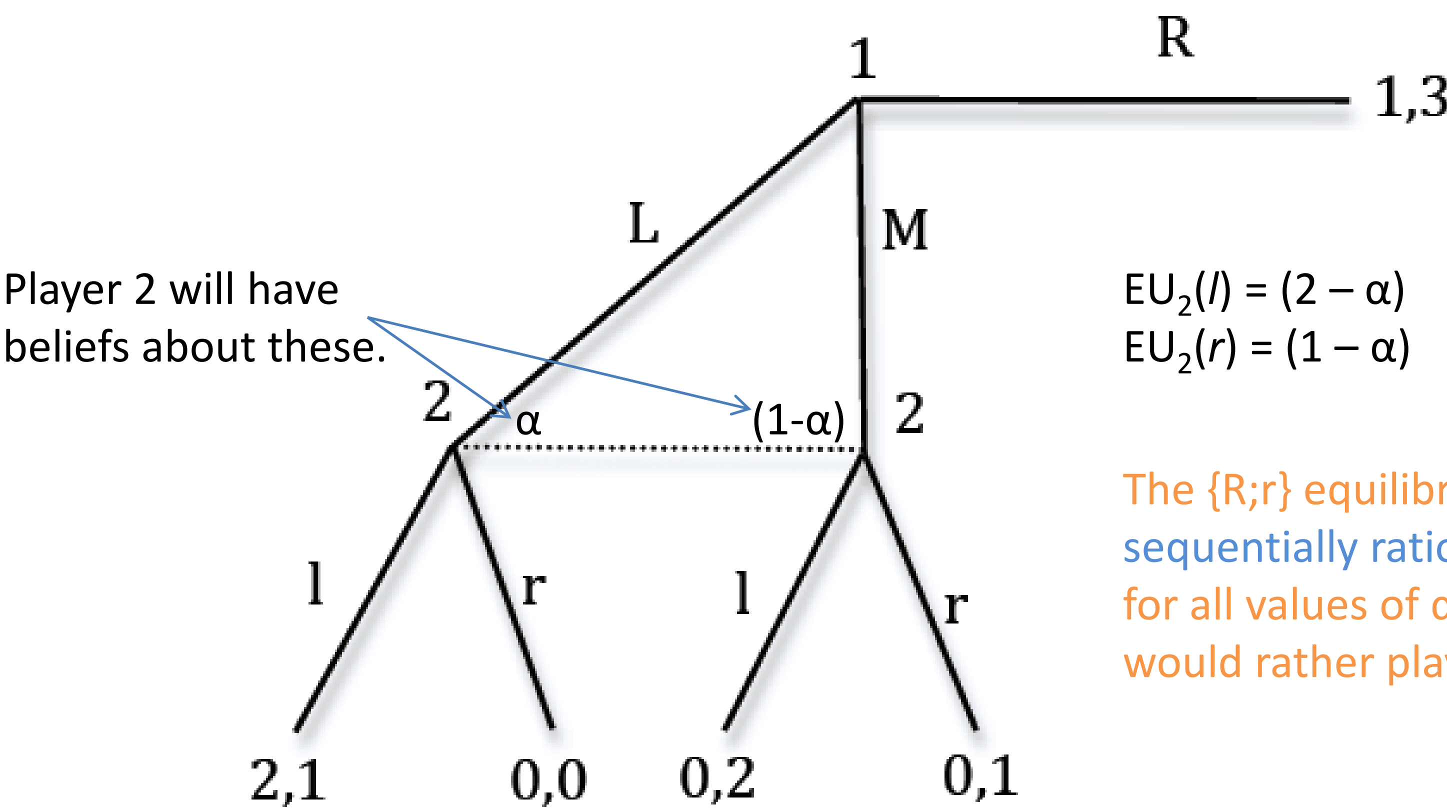
S.E. = {L; (l | α = 1)}

Extensive Form Games with Imperfect Information

Sequential Rationality

- At every information set, the player must “act” optimally in light of her beliefs and the other player’s strategies.

Extensive Form Games with Imperfect Information

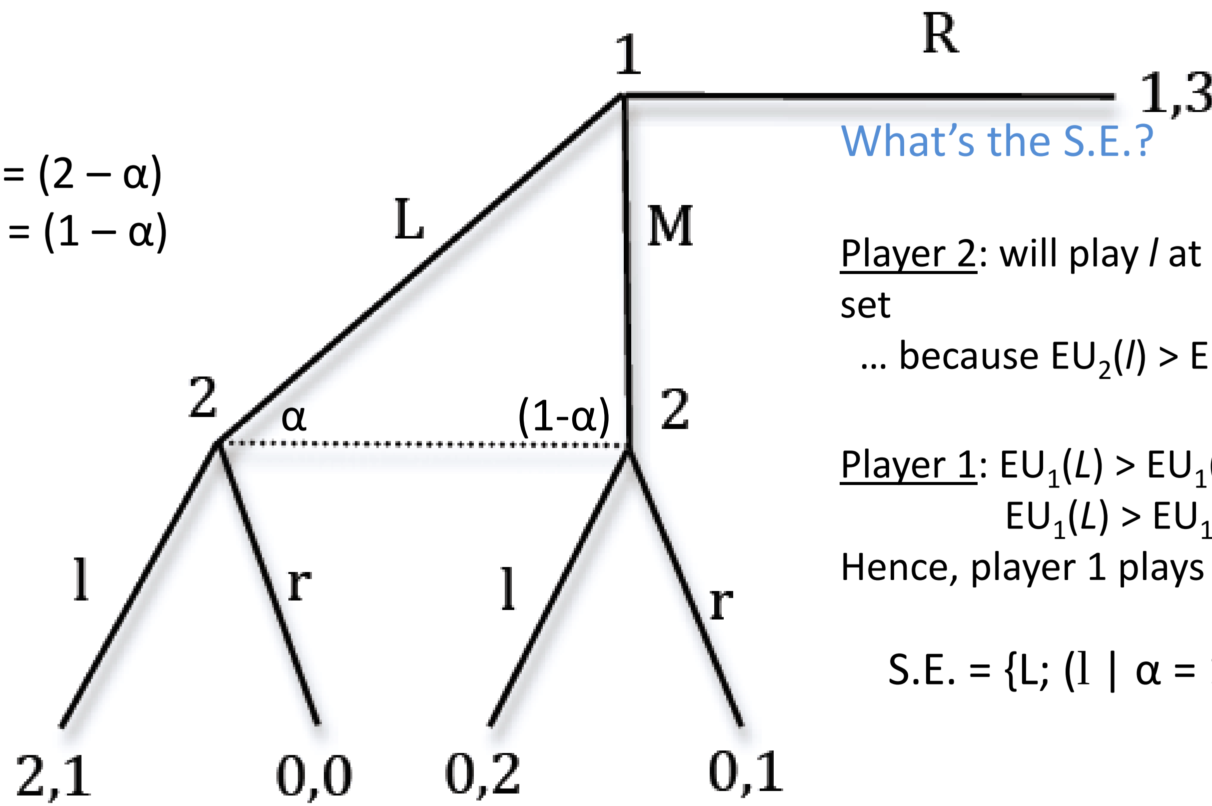


$EU_2(l) = (2 - \alpha)$
$EU_2(r) = (1 - \alpha)$

The {R;r} equilibrium is not sequentially rational …because for all values of α, player 2 would rather play *l*.

Extensive Form Games with Imperfect Information

$EU_2(l) = (2 - \alpha)$
$EU_2(r) = (1 - \alpha)$



What's the S.E.?

<u>Player 2</u>: will play *l* at his information set
… because $EU_2(l) > EU_2(r)$.

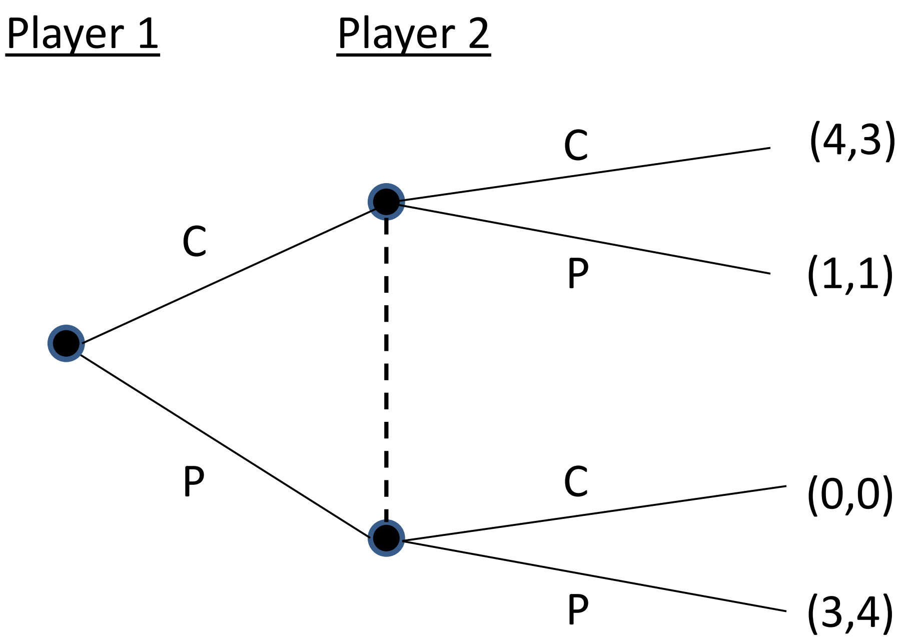
<u>Player 1</u>: $EU_1(L) > EU_1(M)$, and
$EU_1(L) > EU_1(R)$.
Hence, player 1 plays L.

S.E. = $\{L; (l \mid \alpha = 1)\}$

Practice: Battle of the Sexes 2

Two PhD students, using a dating service, decide whether to meet in a pub or a café. They just forgot where to meet.

What are the pure strategy S.E.?



Hint:
1- consider a reasonable strategy profile for both plays

2- make beliefs consistent with that strategy profile.

3- evaluate whether it would be rational for either player to change strategies or beliefs.

Incomplete Information

Imperfect information (what we've been talking about)

- Players are uncertain about at least part of the history of play.
- Modeled using information sets.

Incomplete information

- Players may *not* know all the details of the game they are playing (e.g., the order of moves, the preferences of the other players, ect.).
- *We are mostly interested in cases where at least one player does not know the other player(s) preferences – a type of incomplete information.*

Incomplete information about player(s) preferences are modeled *as if* they are games of imperfect information.

- Before the game is played, Nature randomly determines the players' preferences (from all possible sets of preferences).
- Players know the probability distribution that Nature uses, which is common knowledge.

Incomplete Information

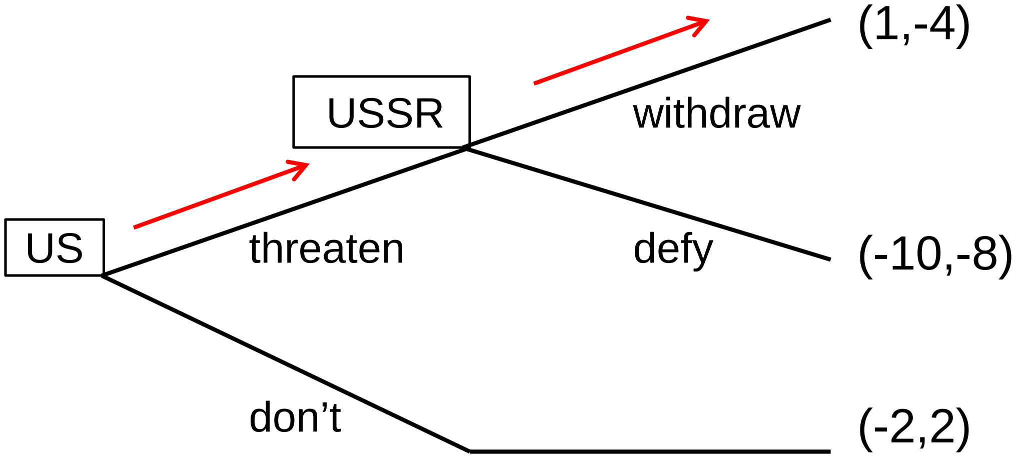
Player Types

- A **player type** is a set of preferences for a player.
  - All possible types for player *i* are called player *i*'s type space.
- Players know their own types (i.e. they know their own preferences) but they are uncertain about the other player(s) type (i.e. the other player's preferences).
- Nevertheless, there is a probability distribution on the other player's types that is common knowledge.

Ex: Cuban Missile Crisis

Two Simple Games (complete information)

1. **Soviet Softies**

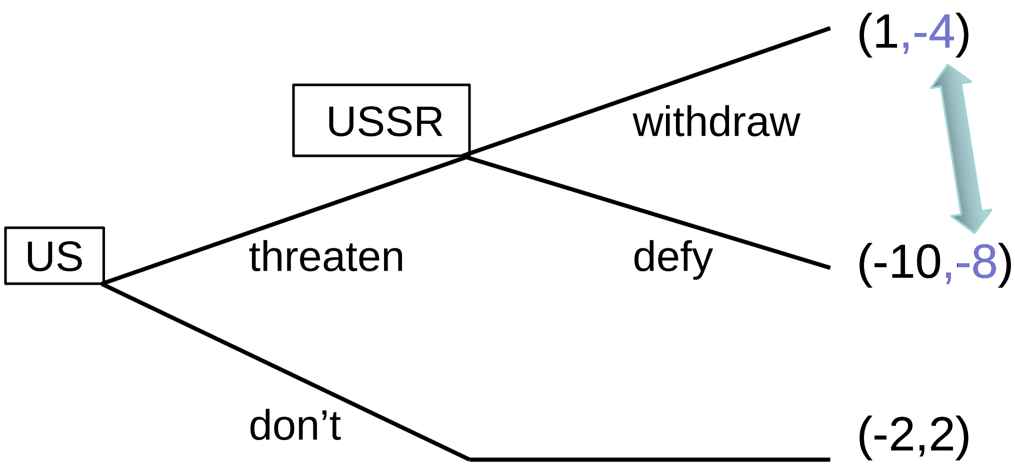


What’s the S.P.E.? S.P.E.= {threaten; withdraw}

Ex: Cuban Missile Crisis

Two Simple Games (complete information)

2. Soviet Hardliners

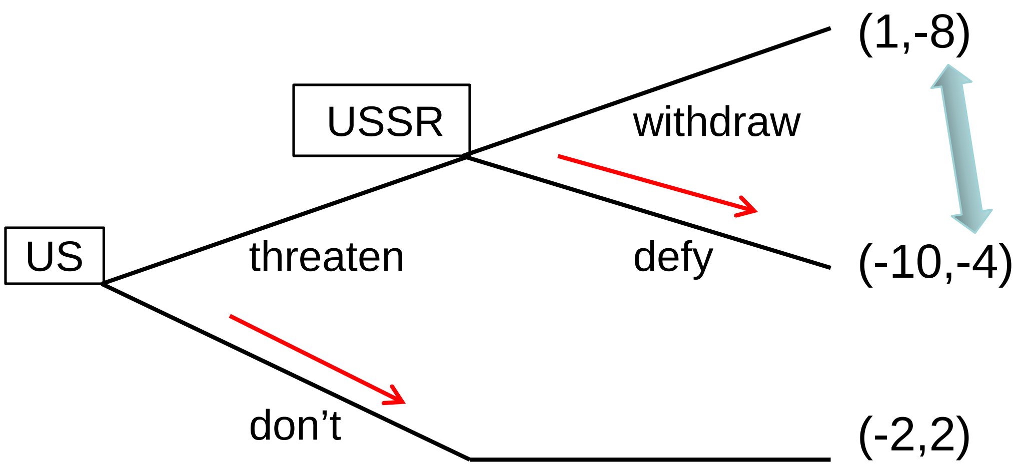


Same game, but a different set of preferences (at least for the USSR).

Ex: Cuban Missile Crisis

Two Simple Games (complete information)

2. Soviet Hardliners

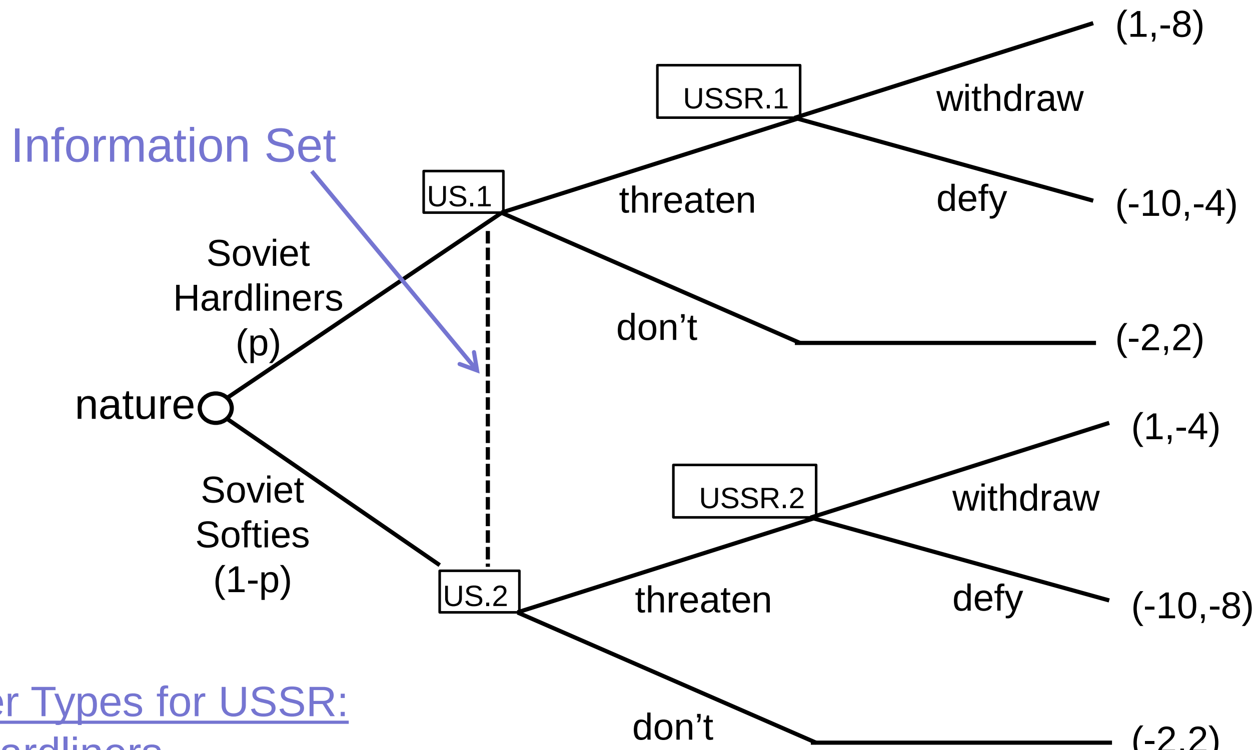


Same game, but a different set of preferences (at least for the USSR).

What's the S.P.E.? S.P.E.= {don't; defy}

Ex: Cuban Missile Crisis

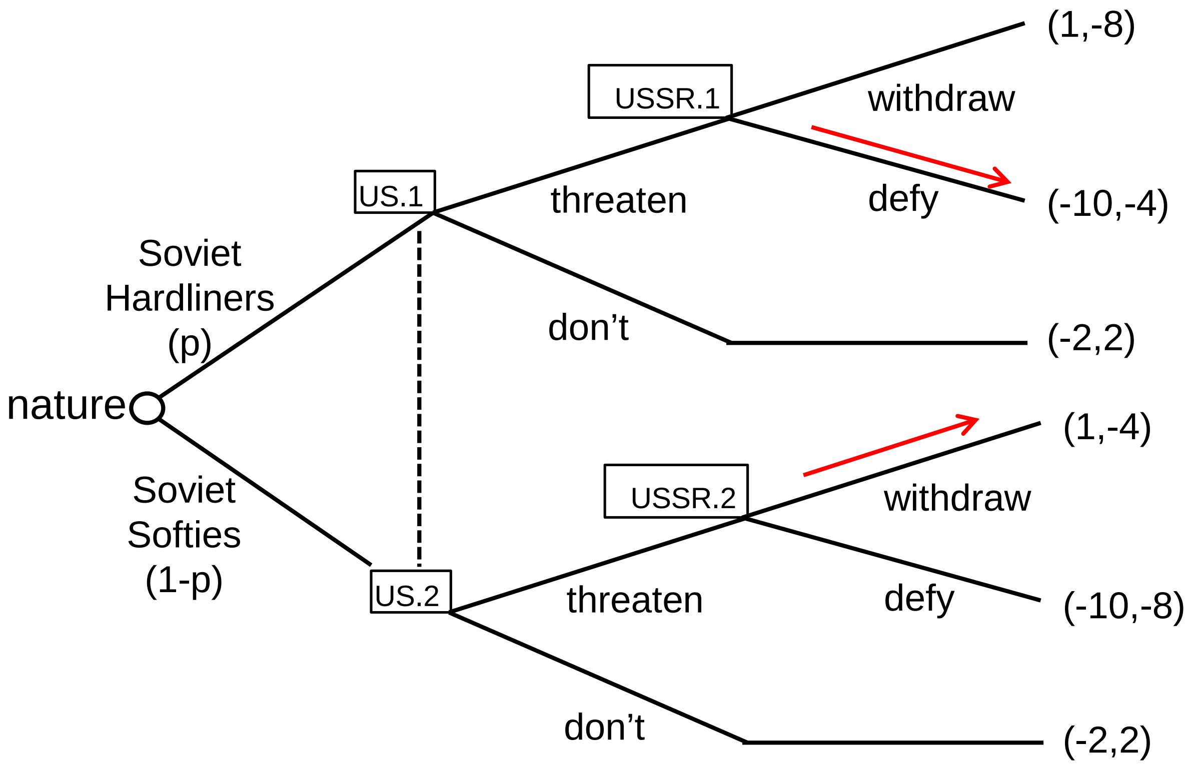
C. Incomplete Information: Unknown Soviet Payoffs



Player Types for USSR:
t_1 = hardliners
t_2 = softies

Ex: Cuban Missile Crisis

C. Incomplete Information: Unknown Soviet Payoffs

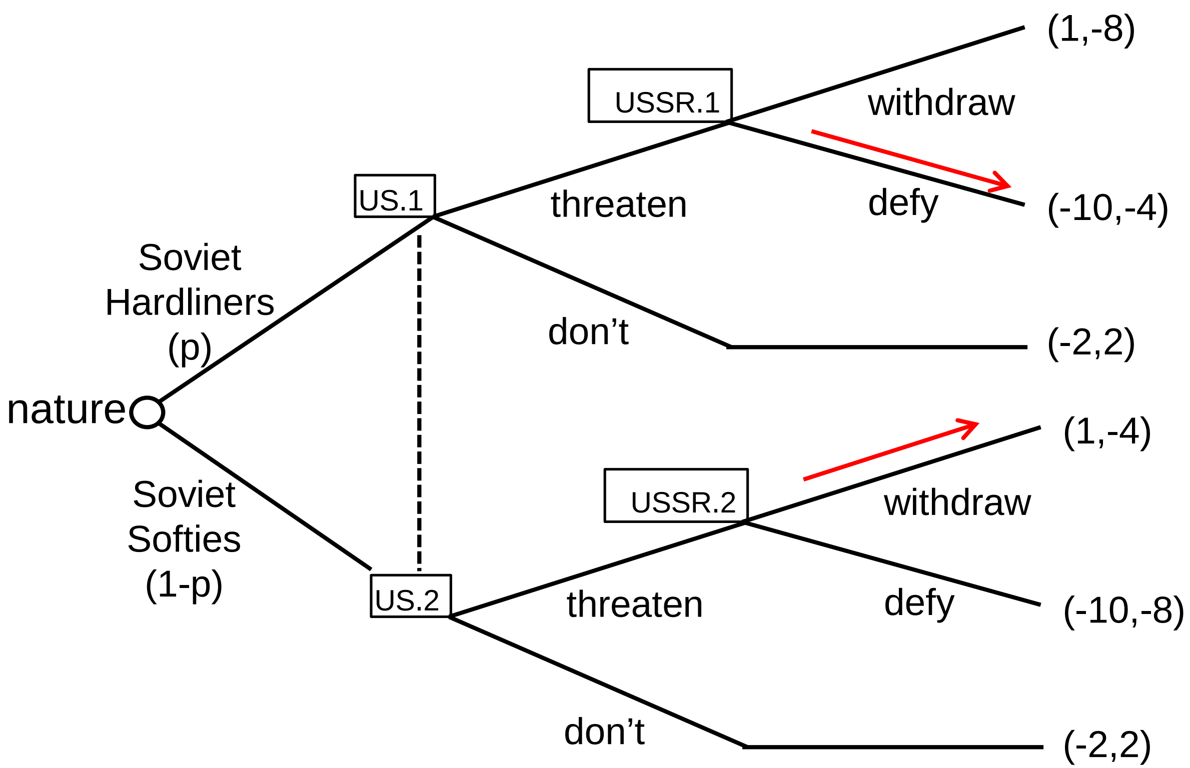


What's the S.E.?

…but the US doesn't know the Soviet type, so it must make a calculation.

Ex: Cuban Missile Crisis

C. Incomplete Information: Unknown Soviet Payoffs



If US threatens:
US payoff is
-10p + 1(1-p)
= 1 – 11p

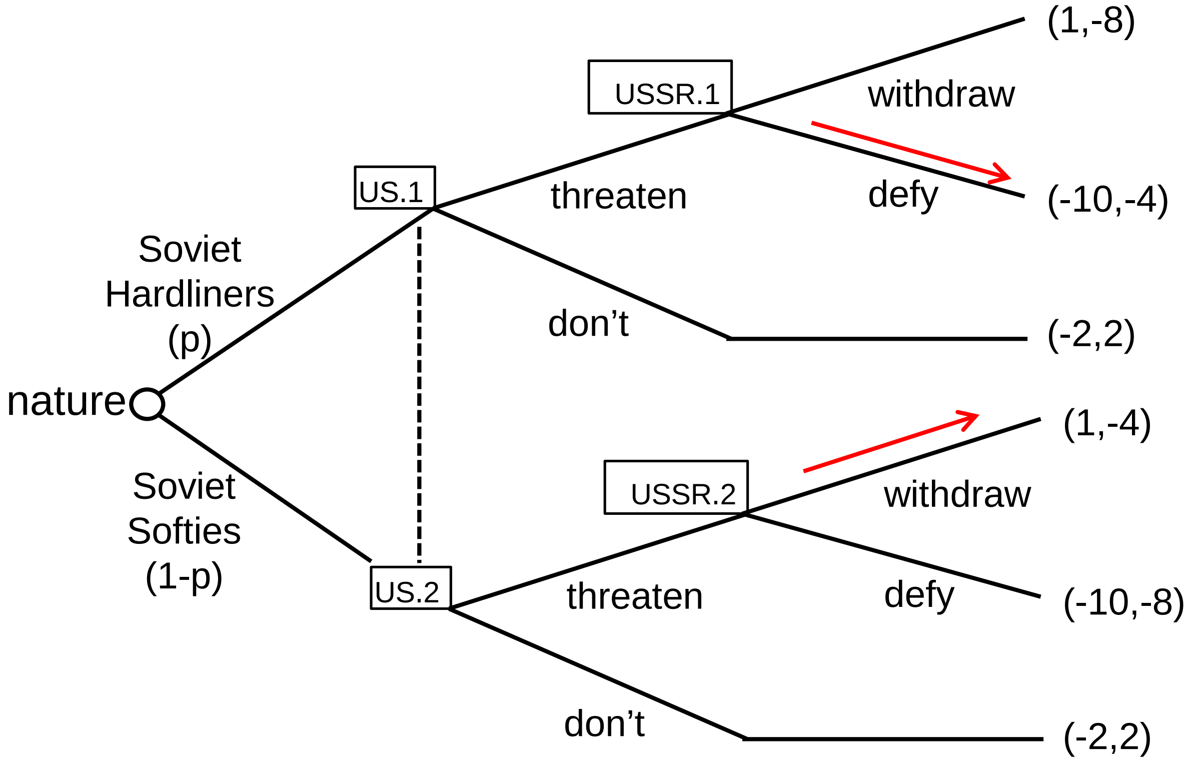
If US don't:
US payoff is
-2p + -2(1-p)
= -2

Hence US threatens iff:
1 – 11p > -2
– 11p > -3
p < 3/11

The US threatens if the probability that the soviets are hardliners is small (prob. less than 3/11).

Ex: Cuban Missile Crisis

C. Incomplete Information: Unknown Soviet Payoffs



<u>If US threatens</u>:
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US payoff is
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= -2

<u>Hence US threatens iff</u>:
1 – 11p > -2
– 11p > -3
p < 3/11

S.E. = {(threaten | p < 3/11); withdraw}.
= {(don't | p > 3/11); defy}.

Subgame Perfect Bayesian Equilibrium (PBE)

- PBE requires **weakly consistent beliefs**
  - Imposes the additional restriction that *beliefs* must be consistent with Bayes' Rule where ever possible (on the path-of-play).
  - This allows players to update their beliefs about another player's type conditional upon arriving at one of their information sets.

- PBE is very similar to Sequential Equilibrium.
  - If each player has at most two types or there are at most two periods of play, PBE and SE will be identical.
  - See Fudenberg and Tirole, 1993, pp. 345-350, for a more complete comparison.

Formal Apparatus

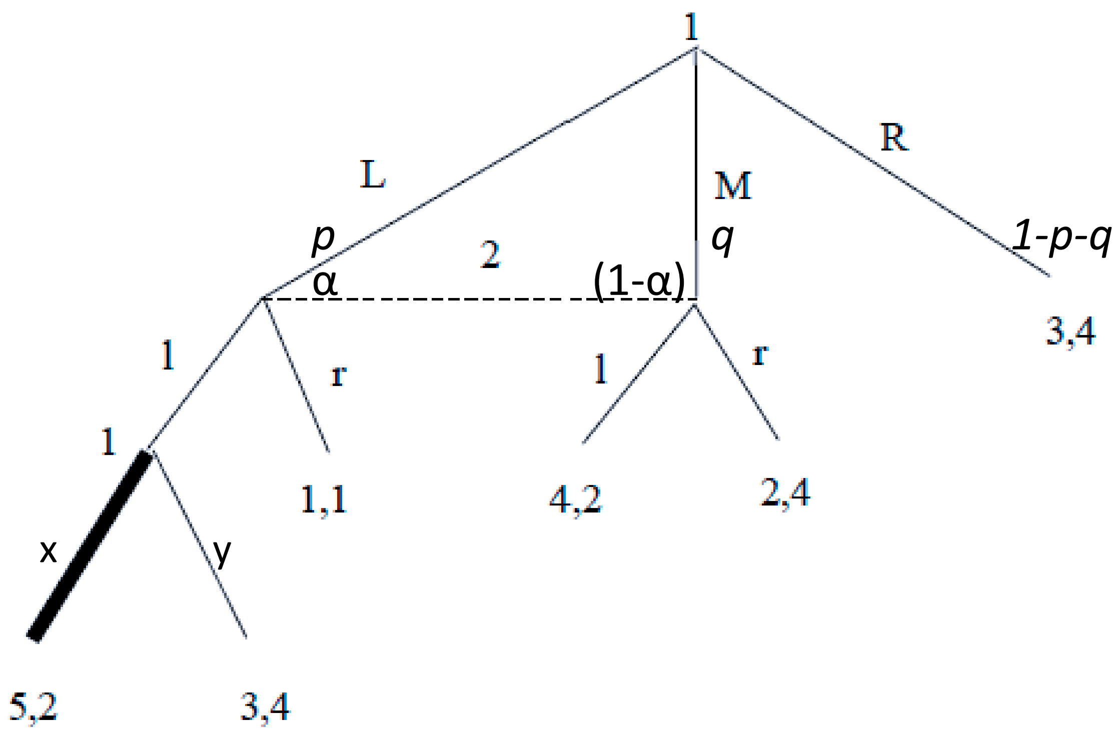
Weak Consistency Requirement

- The consistency requirement demands that along the path of play, player's beliefs are derived from the strategy profile in conjunction with Bayes' rule.
- For any information set I_k that is reached with positive probability, let h^* be a history that is party of I_k, and β the profile of behavioral strategies. Then the player's belief that h^* has occurred is given by

$$Pr(h^*|I_k) = \frac{Pr(h^*|\beta)}{\sum_{h \in I_k} Pr(h|\beta)}.$$

  - This formula comes from a straight forward application of Bayes Rule:

New Example

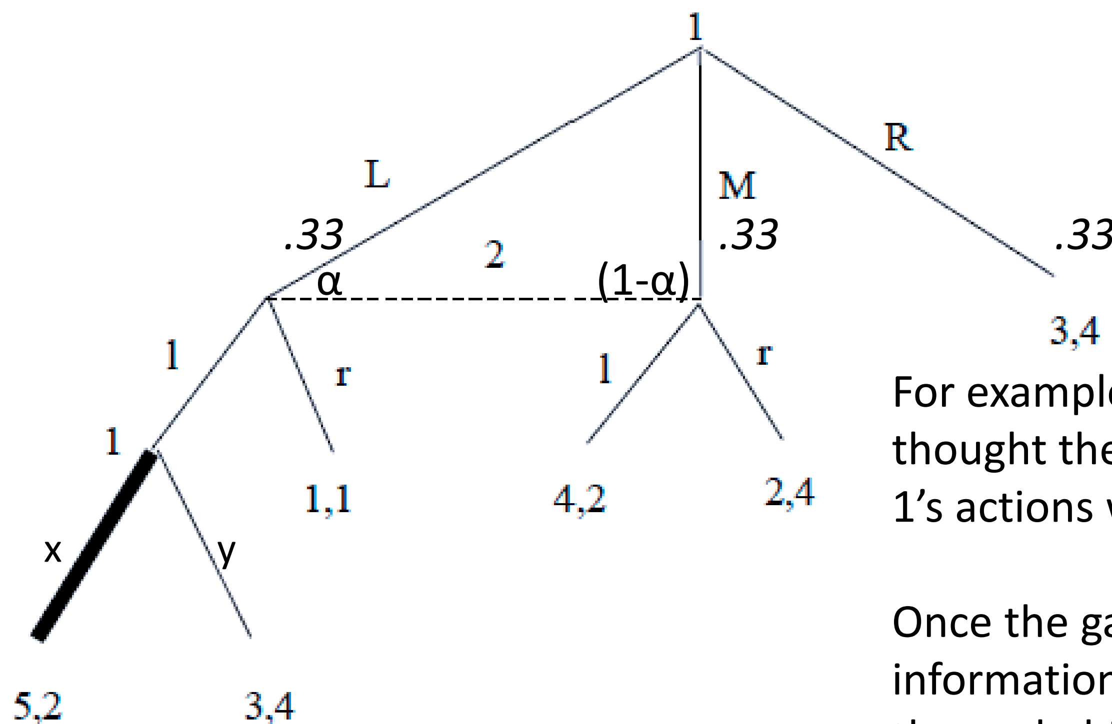


Suppose player 1 moves L with probability p, M with probability q, and R with probability (1-p-q). Then 2's beliefs at his information set must be given by:

$$\begin{aligned} Pr(h=L) &= \alpha &= \frac{p}{p+q} \\ Pr(h=M) &= 1-\alpha &= \frac{q}{p+q} \end{aligned}$$

This is Bayesian updating.

New Example



For example: Suppose 2 initially thought the probability of each of 1's actions were (.33,.33,.33).

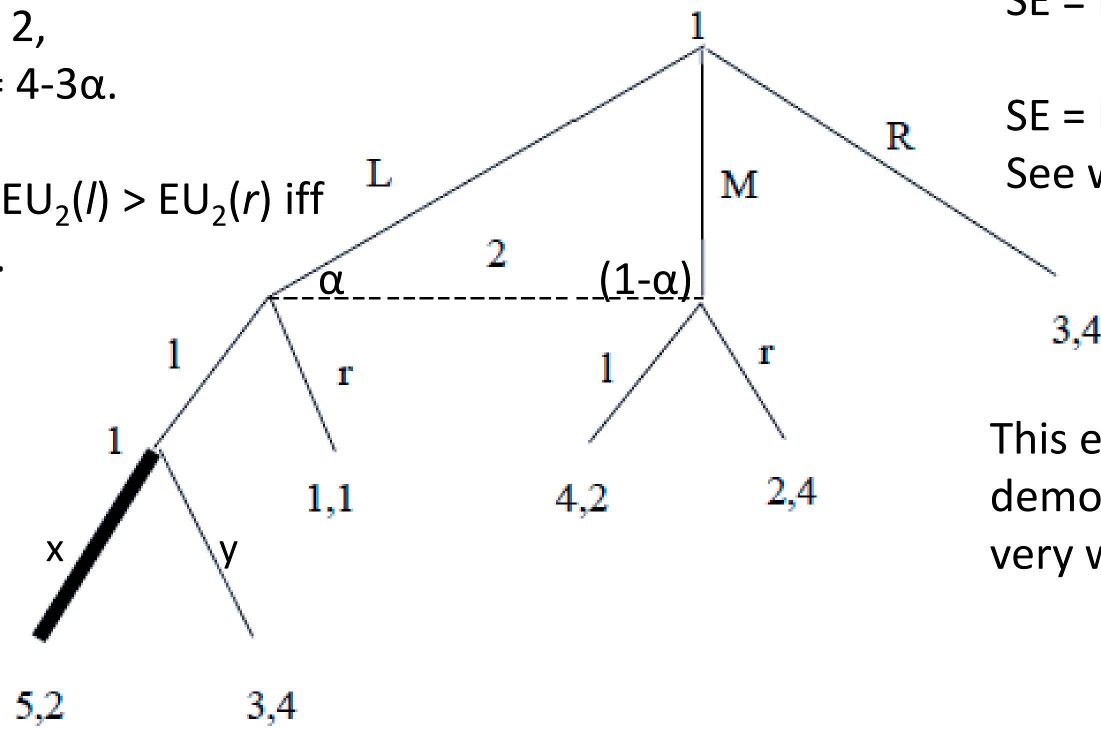
Once the game was at 2's information set, he would update the probability of L to .5 = .33/(.33+.33), and the probability of M to .5 = .33/(.33+.33).

Bayesian updating allows us to correct beliefs given information from the play of the game.

New Example

<u>Solve at 2's info set</u>:
(unnecessary, but let's do it)
$EU_2(l) = 2$,
$EU_2(r) = 4\text{-}3\alpha$.

Hence, $EU_2(l) > EU_2(r)$ iff $\alpha > 2/3$.



Pure Strategy
SE = PBE = {(L,x); *l* | α= 1}.

SE = PBE = {(R,x)); r | α= 0}.
See why?

This example does not demonstrate Bayes Rule very well.

<u>Player 1 (possible strategies):</u>

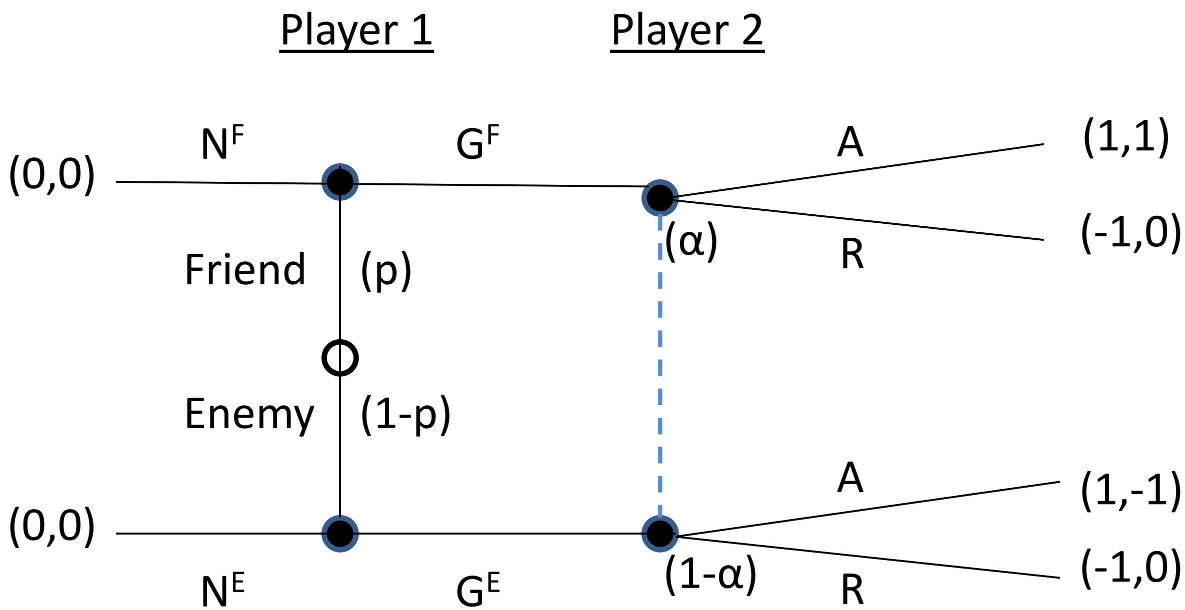
(L, x), then a consistent belief is α = 1, which would make 2 play *l*. Payoff (5,2). No rational deviation for player 1 (or 2). …The beliefs are consistent, because $Pr(h = L) = \alpha = 1$ and $\frac{p}{p+q} = 1$.

(M, x), then a consistent belief is α = 0, which would make 2 play *r*. Payoff (2,4). But player 1 prefers deviating to R, so this is not an equilibrium.

EX: The Gift Game

- Nature determines player 1's type: friend (with probability p) or enemy (with probability 1-p). Player 1 knows her type.
- Player 1 then decides to give a gift to player 2 (which is wrapped). Friends give desirable objects, like cake and CDs, while enemies give undesirable objects like rocks and frogs.
- If player 1 chooses to not give a gift (N), the game ends.
- If player 1 chooses to give a gift (G), then player 2 decides whether to accept (A) or reject (R) the gift.
- Player 2 does not observe player 1's type directly.

EX: The Gift Game



Note: N^F means the strategy "not give a gift" conditional upon being type "friend."

Conditional Beliefs About Types

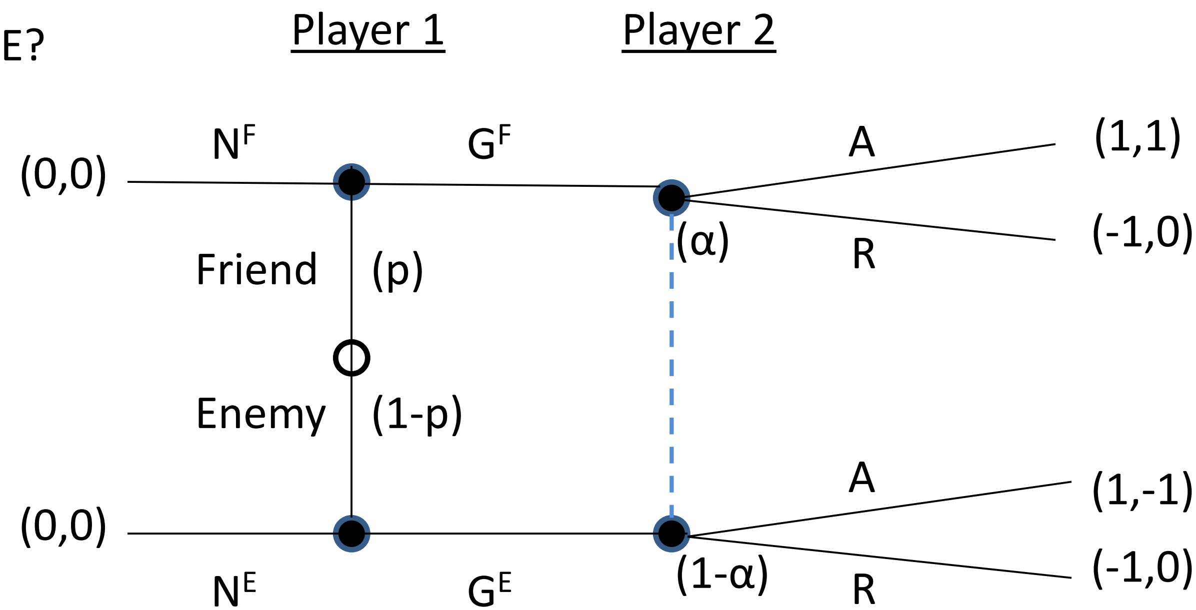
- Probability *p* represent's player 2's initial belief about player 1's type.
- But player 2 might learn something about player 1's type through 1's action. As a result player 2 would have an updated belief about player 1's type.
  - Ex: if player 2 knew that player 1 plays (N^F, G^E) … perhaps because it was the only rational move for player 1… then if player 2 received a gift, he would conclude that 1 must be an enemy.
- In general, player 2 has an updated belief about player 1's type, conditional upon arriving at player 2's information set.
  - In the figure, α is player 2's probabilistic belief that he is at the top node when his information set is reached (i.e. when he has received a gift).
  - (1-α) is player 2's probabilistic belief he is at the bottom node when his information set is reached.

Conditional Beliefs About Types

- Conditional beliefs allow us to evaluate rational behavior at all information sets, even those that may not be reached in equilibrium.
  - Suppose player 1 plays N^F, N^E (neither type gives a gift) and player 2 knows it. In this case α represents player 2's belief about player 1's type off the equilibrium path (not type and play).

EX: The Gift Game

What is the PBE?



The following prep work will be useful in finding the PBE:

$EU_2(A) = \alpha + (-1)(1- \alpha)$

$EU_2(R) = 0.$

At his information set, player 2 selects A iff $EU_2(A) > EU_2(R)$

$$2\alpha - 1 > 0$$

$$\alpha > ½.$$

He will select R if $\alpha < ½$.

Perfect Bayesian Equilibrium

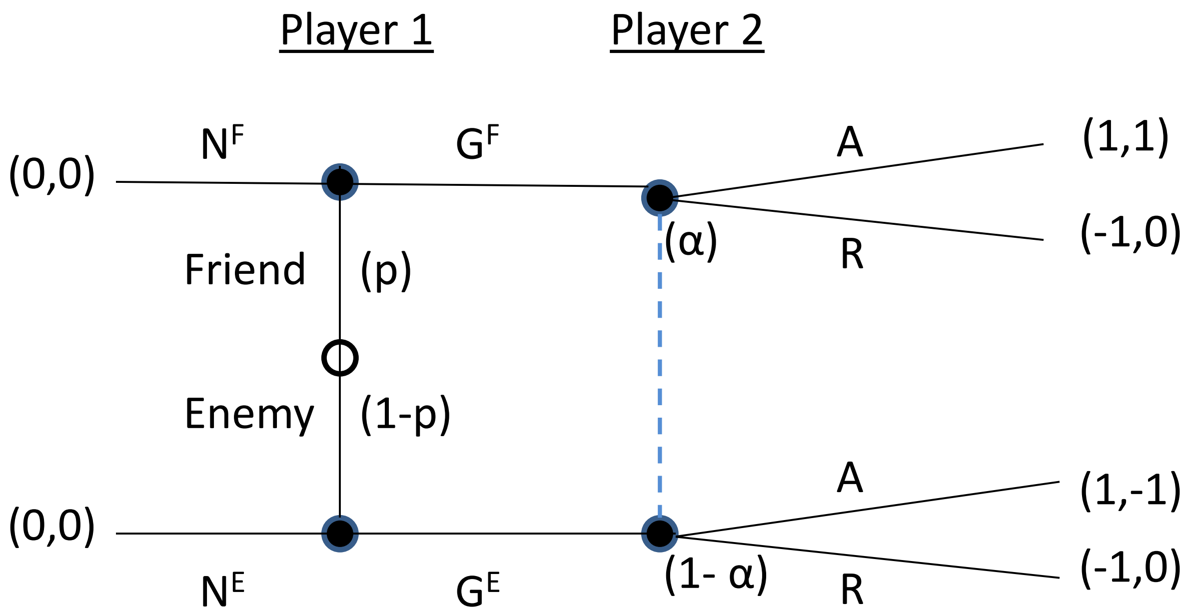
- With uncertainty about player types, two types of PBE:
  - An equilibrium is separating if the types of a player behave differently.
  - An equilibrium is pooling if the types behave the same.
  - An equilibrium is semi-pooling/semi-separating if you get a bit of both.

- To determine the set of pure strategy PBE for the game:
  1. Start with a strategy for player 1 (separating or pooling).
  2. If possible, calculate α by Bayes rule.
     - If you can't use Bayes rule, arbitrarily select α and check whether its value against the next steps of the procedure.
  3. Given α, calculate player 2's optimal action.
  4. Check whether player 1's strategy is a best response to player 2's strategy. If so, you have a BPE.

Step 2:

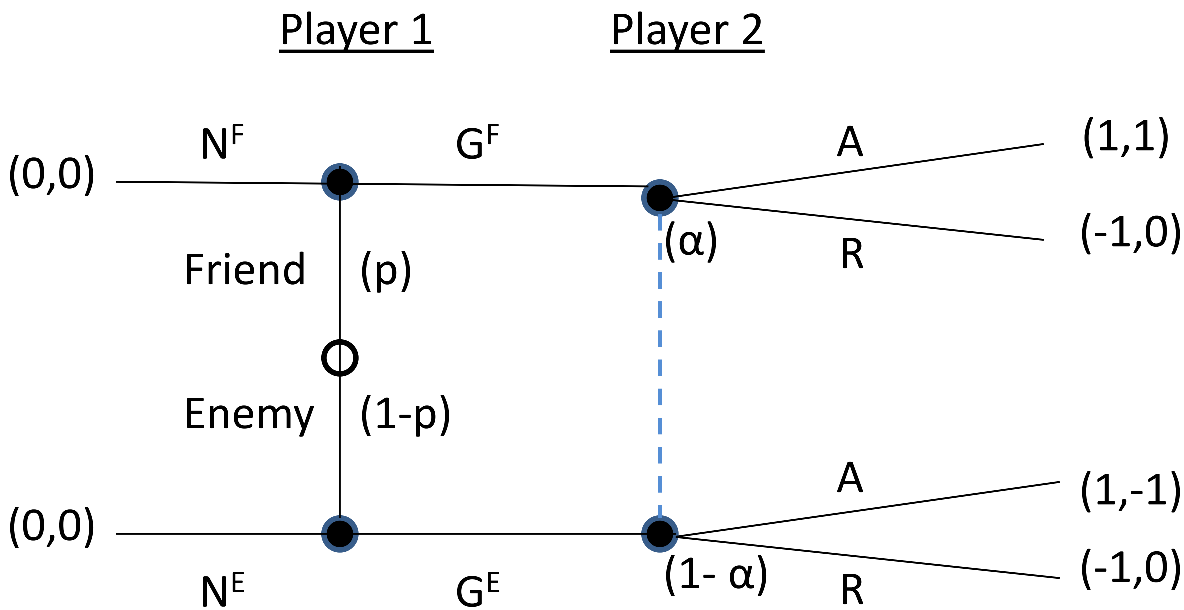
$$\alpha = Prob[F|G] = \frac{Prob[G|F]Prob[F]}{Prob[G]} = \frac{(0)p}{Prob[G]}$$

EX: The Gift Game, PBE



Separating with $N^F G^E$: Given this strategy for player 1, it must be that α = 0 (step 2). Thus, player 2's optimal strategy is R (step 3). But then player 1 would strictly prefer not to play G^E when of the enemy type (step 4). Therefore, there is no PBE in which $N^F G^E$ is played.

EX: The Gift Game, PBE

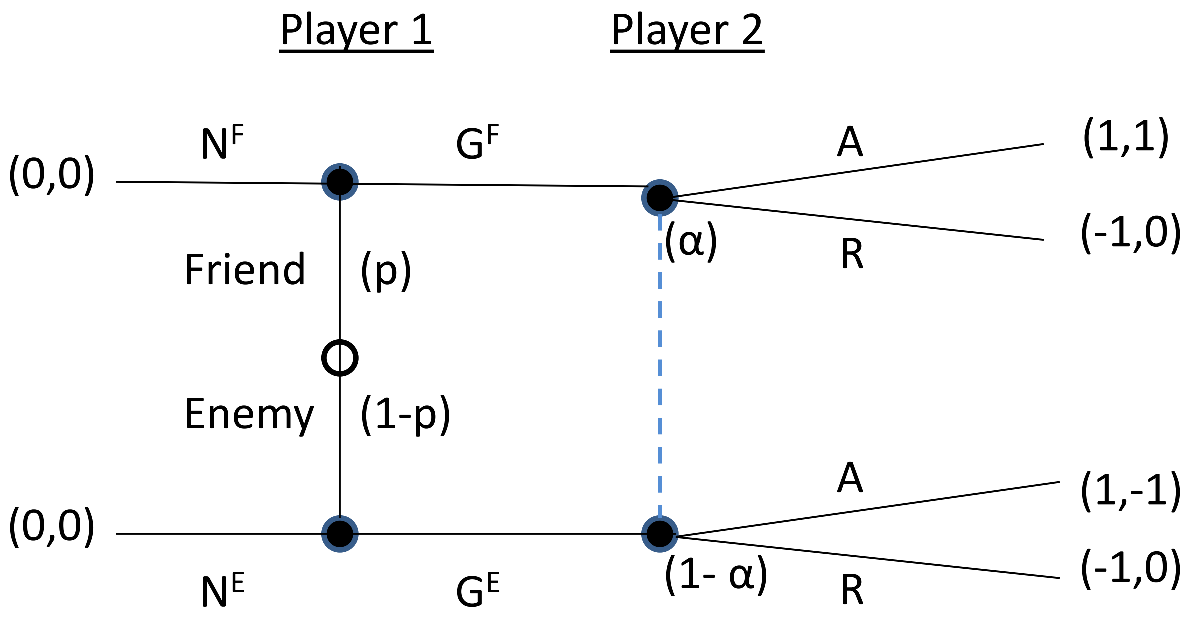


Separating with $G^F N^E$: Given this strategy for player 1, it must be that $\alpha = 1$ (step 2). Thus, player 2's optimal strategy is A (step 3). But then the enemy type of player 1 would strictly prefer to play G^E rather than N^E (step 4). Therefore, there is no PBE in which $G^F N^E$ is played.

Step 2:

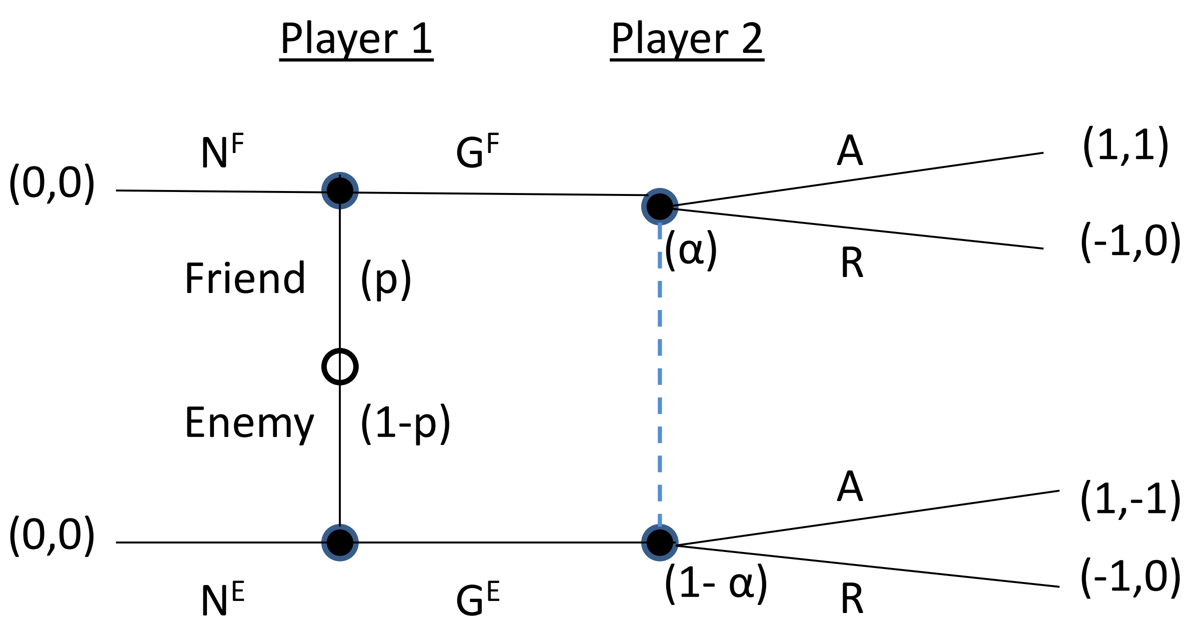
$$\alpha = Prob[F|G] = \frac{Prob[G|F]Prob[F]}{Prob[G]} = \frac{(1)p}{1} = p$$

EX: The Gift Game, PBE



Pooling with $G^F G^E$: Here, Bayes' rule requires $\alpha = p$, so player 2 optimally selects A if and only if $p \geq ½$ (remember prep work). In the event that $p < ½$, player 2 must select R, in which case neither type of player 1 wishes to play G in the first place. Thus, there is no PBE of this type when $p < ½$. When $p \geq ½$ there is a PBE in which $\alpha = p$ and $\{G^F G^E; A\}$ is played.

EX: The Gift Game, PBE



Pooling with N^FN^E: Bayes' rule does not determine α. But notice that both types of player 1 will continue not giving gifts only if player 2 selects R. Strategy R is optimal as long as $\alpha \leq ½$. Thus, for every $\alpha \leq ½$, there is a PBE in which player 2's belief is α and the strategy profile $\{N^FN^E; R\}$ is played.

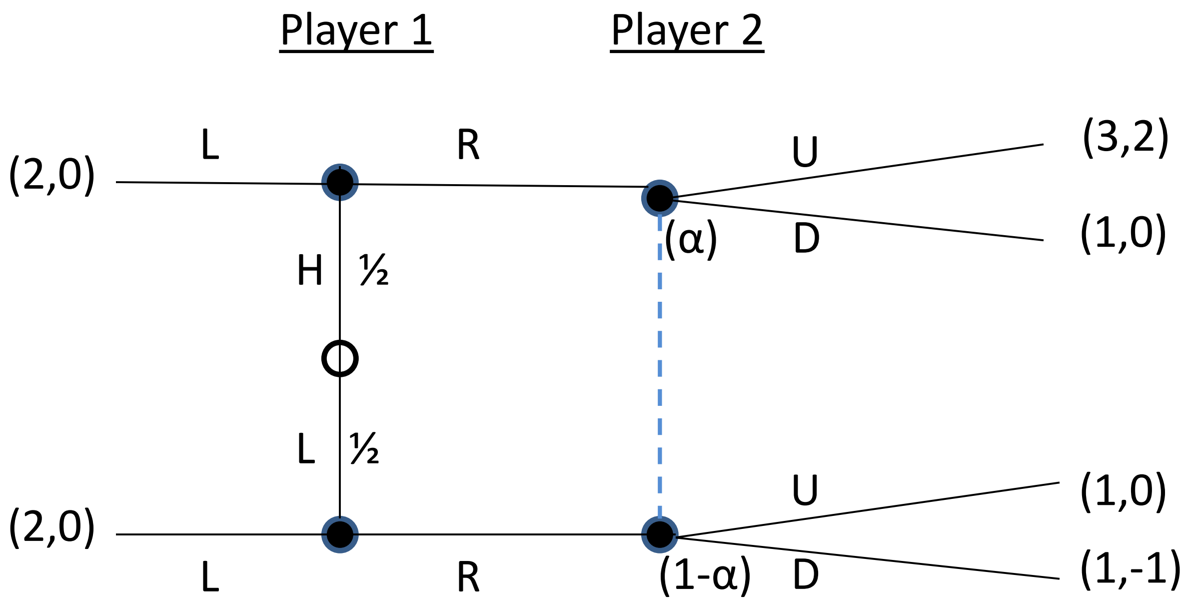
Summary of the Gift Game

- Because both types of player 1 have the same preferences over the outcomes, there is no separating equilibrium.

- There is always a pooling equilibrium in which no gifts are given because player 2 believes receiving a gift signals an enemy.

- If there is a great enough chance of encountering a friend ($p \geq ½$), then there is a pooling equilibrium in which gifts are given by both types. In this equilibrium a sanguine player 2 gladly accepts the gift.

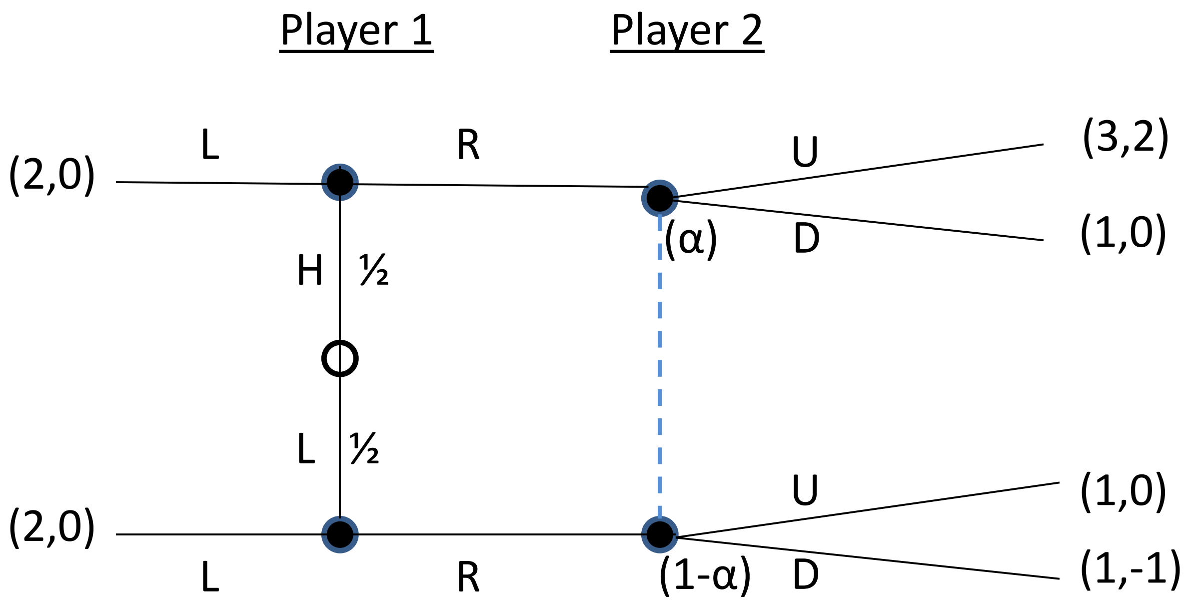
Practice



1. Does this game have a *separating* perfect Bayesian equilibrium? If so, fully describe it.

2. Does this game have a *pooling* perfect Bayesian equilibrium? If so, fully describe it.

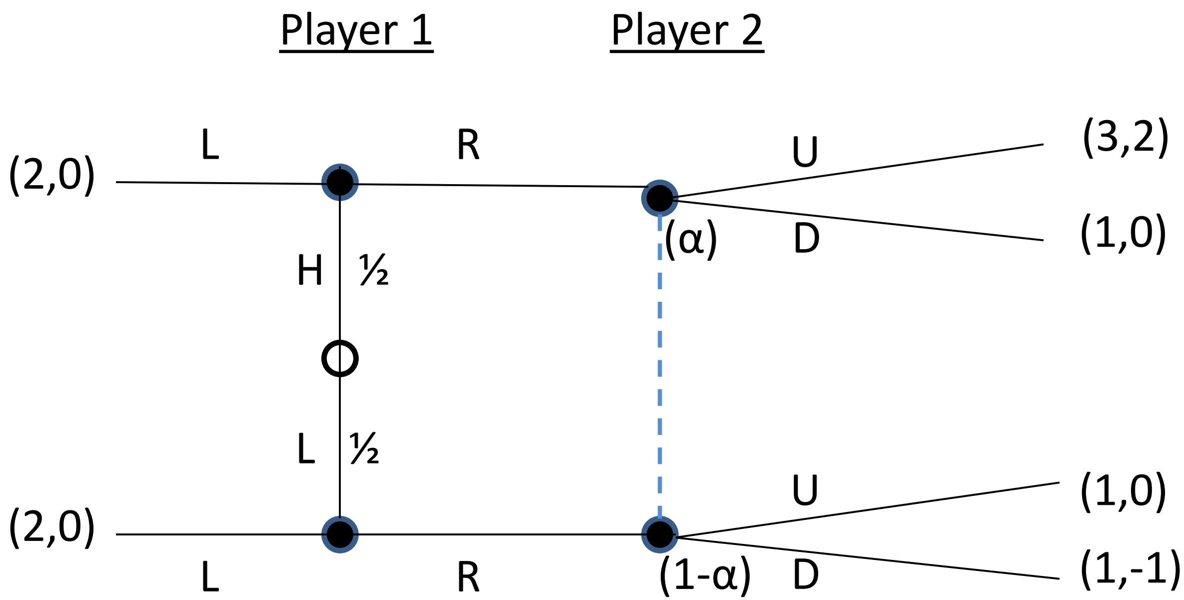
Practice



Separating with $L^H R^L$: given this strategy for player 1, it must be that α = 0. Thus, player 2's optimal strategy is U. But then player 1 would strictly prefer L^L to R^L. Therefore, there is no PBE in which $L^H R^L$ is played.

Separating with $R^H L^L$: given this strategy for player 1, it must be that α = 1. Thus, player 2's optimal strategy is U. Because the high type of player 1 cannot gain from deviating to L^H (giving her 2 rather than 3) and the low type cannot gain from deviating to R^L(giving her 1 rather than 2), ($R^H L^L$, U) and α = 1 is a PBE.

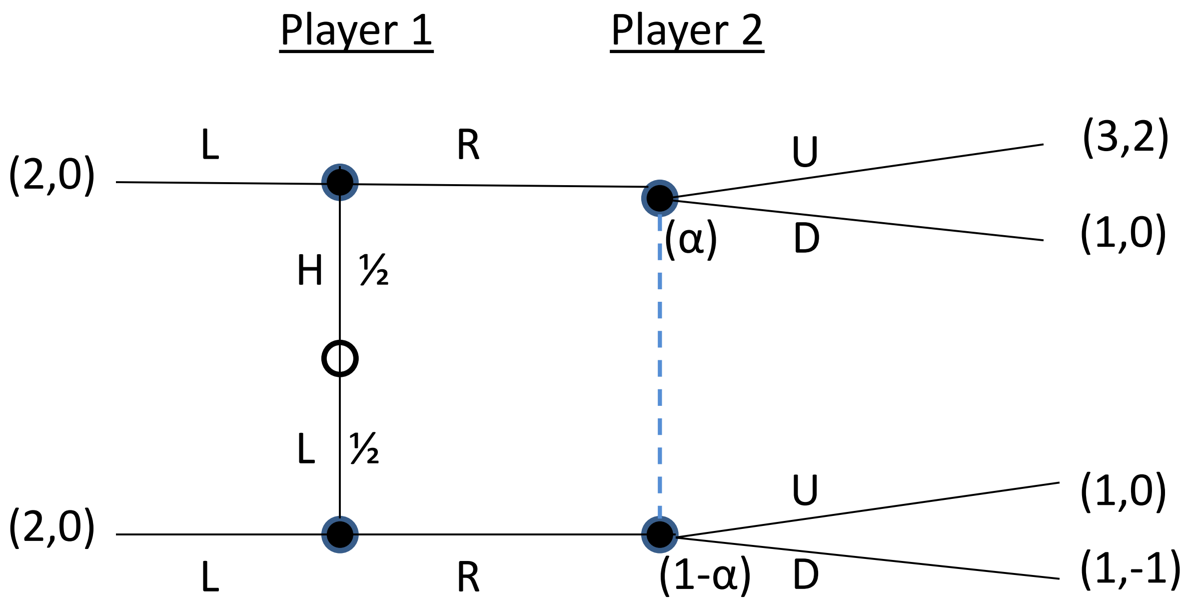
Practice



Pooling with $L^H L^L$: Bayes' rule does not determine α. But notice that player 1 will prefer to deviate to $R^H L^L$ if player 2 plays U. Further notice that player 2 always prefers U to D. Hence, the deviation is rational and there is not a PBE in which $L^H L^L$ is played.

Pooling with $R^H R^L$: Bayes' rule requires α = ½. However, it is rational for player 1 to deviate to $R^H L^L$, because 2>1, so there is not a PBE in which $R^H R^L$ is played.

Practice

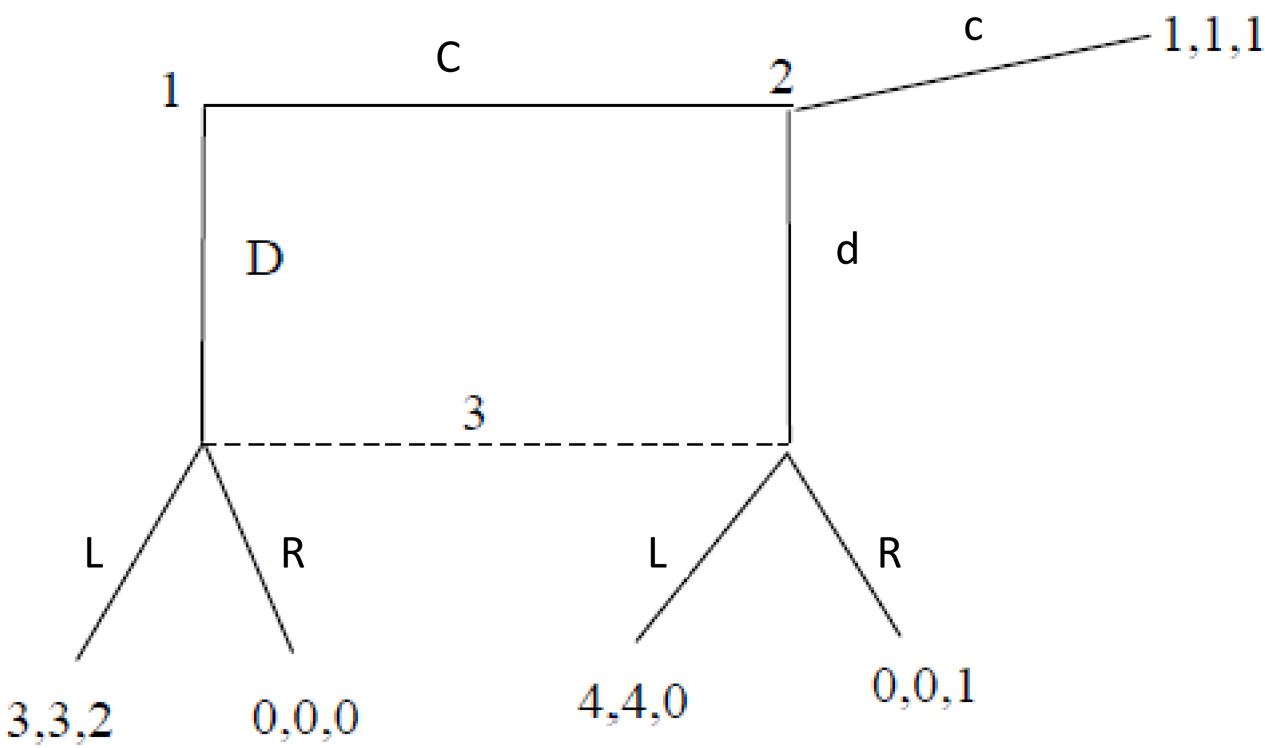


Remarks: There are no pooling equilibria. There is only a separating equilibria where player 1 plays right if the high type and left if the low type, while player 2 plays up.

This is partly due to the fact that player 2 always prefers U to D, which player 1 takes advantage of by having each of her types play differently.

Reinhard Selten

Reinhard Selten's Horse



Extra Credit Game (HW 5)

- Because this stuff is so bloody hard, I'm going to give 5 extra credit points on HW5 to everyone who showed up today.
  - Note to self: pass attendance sheet.