The Spatial Error Model
Another Approach for Areal Data Using Maximum Likelihood

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Spatial Data Analysis
Objectives

By the end of this meeting, participants should be able to:

- Estimate, using maximum likelihood, a regression model with a spatial error term.
- Visually present areal data using maps.
Again, start with: $y_i = x_i \beta + \varepsilon_i$.

This time we say: $\varepsilon_i = \lambda w_i \xi_i + \epsilon_i$.

By substitution, the full spatial error model is:

$$y_i = x_i \beta + \lambda w_i \xi_i + \epsilon_i.$$  

In matrix notation, this gives us:

$$Y = X\beta + \lambda W \xi + \epsilon,$$

$$\epsilon \sim \mathcal{N}(0, \sigma^2 I).$$
Recall: When estimating a linear model with MLE and the Gauss-Markov assumptions are true, our log likelihood function is:

$$\ln L(\beta, \sigma^2) = -\frac{N}{2} \ln(2\pi) - \frac{N}{2} \ln(\sigma^2) - \frac{(y - X\beta)'(y - X\beta)}{2\sigma^2}.$$  

Our log likelihood function for the spatial error model is:

$$\ln L(\beta, \sigma^2, \lambda) = \ln |I - \lambda W| - \frac{N}{2} \ln(2\pi) - \frac{N}{2} \ln(\sigma^2) - \frac{(y - X\beta)'(I - \lambda W)'(I - \lambda W)(y - X\beta)}{2\sigma^2}.$$  

Which simplifies to:  

$$\ln L(\beta, \sigma^2, \lambda) = \ln |I - \lambda W| - \frac{N}{2} \ln(2\pi) - \frac{N}{2} \ln(\sigma^2) - \frac{(y - \lambda W y - X\beta + \lambda W X\beta)'(y - \lambda W y - X\beta + \lambda W X\beta)}{2\sigma^2}.$$  

We use Ord’s (1975) trick again: Find the eigenvalues of \( W \), \( (\omega_1, \ldots, \omega_n) \). This gives us the determinant we need:

$$|I - \lambda W| = \prod_{i=1}^{n} (1 - \lambda \omega_i).$$
Comparing Three Models of Democracy as a function of GDP

Source: Ward & Gleditsch 2008, Table 3.1

<table>
<thead>
<tr>
<th></th>
<th>Naïve OLS</th>
<th>Lagged DV</th>
<th>Spatial error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Est.</td>
<td>S.E.</td>
<td>Est.</td>
</tr>
<tr>
<td>Intercept</td>
<td>-9.69</td>
<td>2.43</td>
<td>-6.20</td>
</tr>
<tr>
<td>ln p.c. GDP</td>
<td>1.68</td>
<td>0.31</td>
<td>0.99</td>
</tr>
<tr>
<td>( \hat{\rho} )</td>
<td>—</td>
<td>—</td>
<td>0.56</td>
</tr>
<tr>
<td>( \hat{\lambda} )</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Log Lik.</td>
<td>-513.62</td>
<td>—</td>
<td>-491.10</td>
</tr>
</tbody>
</table>

\( n = 158 \)
Choosing Between the Spatial Lag and Spatial Error Models

- Like time series, panel data, and others, there are many ways to fit a model to spatial data.
- Choosing which is a function of assumptions you believe.
- Which is more likely for your data?
  - Is one observation’s value of the dependent variable shaped by neighbors’ values of the dependent variable?
    - Likely answer: Spatially lagged dependent variable model.
  - Is there a lurking variable in the error term that is likely to be similar among neighbors?
    - Likely answer: Spatial error model.
- Empirically, a true spatial error process and a true autoregressive process are hard to untangle.
- Theory should dominate your view of which model is better, though.
Graphing Areal Data

- It is always good practice to visualize your data.
- Maps of point or lattice observations convey substantial information.
  - What patterns are apparent between variables? (See: V.O. Key.)
  - Are there trends in the data? Is there clustering?
- Usually shading or color saturation is how we convey values of variables on maps.
- Unfortunately, shading and color is pretty far down the list in humans’ perceptual accuracy (Cleveland & McGill 1984).
- At least do the best you can with color:
  - Choose two colors, set white as the middle value, and create a bipolar color gradient.
  - You may have to redraw in grayscale for print, but many journals now allow color graphics for the online edition.
- Note: Accurate geographic location is important and conveys a lot of information, too. I personally do not like the idea of resizing areal units based on the value of an area.
  - Consider: Does doing this affect neighbor connections? Does it affect distance between two points?
For Next Time

- Read §2.1-2.3 from Banerjee, Carlin, & Gelfand.
- Subset the data only to the state of Georgia (`state`).
- Estimate a naïve OLS regression of Hispanic registration in 2004 (`hreg04`) as a function of Hispanic registration in 2002 (`hreg02`) and the Hispanic growth rate (`growthrate`).
- Plot a map of the residuals by county. Based on this graph, do the residuals appear to be spatially correlated? Why or why not?
- Choose a spatial weighting scheme and describe your choice.
- Report the results of Geary’s $C$ for the residuals. How do these results compare to your visual analysis?
- Re-estimate your model using a spatially lagged dependent variable and again using a spatial error term. Report all three models in a neat table.
- Which of these models do you believe to be the most sound theoretically? Why?
- Present your results in professional tables and figures. Attach your R code to the back of your final copy.