THEORY: NASH EQUILIBRIUM
Prisoner’s Dilemma

The Story

• Two prisoners held in separate rooms.
• Authorities offer a reduced sentence to each prisoner if he rats out his friend.
• If a prisoner is ratted out by the other guy, then he receives a harsh sentence. If he rats out the other guy, then he receives a lighter sentence.
Matrix representation

<table>
<thead>
<tr>
<th></th>
<th>Player 2 (Column)</th>
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<tbody>
<tr>
<td></td>
<td>C</td>
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<tr>
<td>C</td>
<td>2, 2</td>
</tr>
<tr>
<td>D</td>
<td>5, 0</td>
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</tbody>
</table>

Numbers are utility, not years. Hence, larger numbers are more preferred.

I need a volunteer
A **strategic (or normal form) game** consists of:

- A set of **players**, denoted by \( N = \{1,\ldots,n\} \),
- For every player \( i \in N \), a set of **strategies** \( S_i = \{s_1, s_2, \ldots, s_k\} \),
- The set of **strategy profiles** \( S = \prod S_i \), which are the possible outcomes of the game,
- For every player, preferences over the set of strategy profiles: \( u_i : S \rightarrow \mathbb{R} \).
Matrix representation

Set of Players: \{player 1, player 2\}.

Player 1 (Row)

Set of Strategies for player 1: \{C, D\}.

Set of Strategies Profiles:
\{(C,C),(C,D),(D,C),(D,D)\}

Player 2 (Column)

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<tr>
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<tbody>
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<td>2,2</td>
<td>0,5</td>
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<tr>
<td>D</td>
<td>5,0</td>
<td>1,1</td>
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</tbody>
</table>

Player 1’s preferences for the strategy profiles (in utility).
Model of a strategic game

More on strategy profiles:
• For any strategy profile $s \in S'$, we will use the notation $s_{-i}$ to denote the strategies adopted by all players except player $i$, that is $s = (s_i, s_{-i})$.

Utility Functions:
• A utility function for player $i$ is a function $u_i : S' \rightarrow \mathbb{R}$ the represents the player’s preferences over the strategy profiles.
Common Knowledge

Definition:
• A fact is **common knowledge** if all of the players know it, and all of the players know that all of the players know it, etc.

Rationality:
• We assume that it is common knowledge among all players that they are rational.
  – That is, they want to get the outcomes they prefer most among the ones they can actually attain.
Remarks

Time

• Time is absent from strategic form games. Players cannot make their actions contingent on the actions of other players, perhaps because
  – Players act simultaneously,
  – Players are not informed about the previous moves of the other players.

Information

• We are currently assuming players have complete information. They know the structure of the game, the strategies available, and the preferences of all players.
  – The last assumption will be relaxed when we get to Bayesian Subgame Perfect Equilibrium.
Extra credit on HW2. I’m going to randomly match you with someone in the room. Imagine you are row. How would you play this game?

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<td>Player 1 (Row)</td>
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Chicken

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<td>2 , 2</td>
<td>1 , 3</td>
</tr>
<tr>
<td><strong>H</strong></td>
<td>3 , 1</td>
<td>-1 , -1</td>
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</table>

Player 1 (Row) | Player 2 (Column)

How would two rational actors play this game?
Solution Concepts

Definition

• A **solution concept** is a tool for making a prediction about how rational players are going to play a game. It identifies some strategy profile as more plausible than others.

Desirable Properties of Solution Concepts

• **Existence**: the concept should apply to a wide class of games.

• **Exclusivity**: the concept should narrow down the list of strategy profiles.

• **Robustness**: small changes in the game should not affect the prediction made by the solution concept.
Nash equilibrium

Informal definition
• A Nash equilibrium (NE) is a strategy profile such that no player has a unilateral incentive to “deviate” (if the strategies of all the other players are held constant, no player would like to change his/her strategy).

Formal definition
• A strategy profile \( s^* \in S \) is a Nash Equilibrium in a strategic form game \( G \) if and only if \( \forall i \in N \) and \( \forall s_i \in S_i \),
\[
u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).
\]
Example: PD

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<tr>
<td><strong>D</strong></td>
<td>5, 0</td>
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\{D;D\} is a Nash Equilibrium in this game, because neither player has a unilateral incentive to deviate.
Remarks

• Indifference keeps a player in equilibrium. In order to have an “incentive to deviate” a player’s utility from another action must be strictly better than s*.

• A Nash equilibrium is a “stable” outcome in the sense that it is self-enforcing.

• There may be multiple Nash equilibria in a game.

• Nash equilibria are not necessarily efficient. All players may unanimously prefer another outcome to a Nash Equilibrium.
Best Response Function

Definition

- The best response function for player $i$ is defined by

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(\tilde{s}_i, s_{-i}) \forall \tilde{s}_i \in S_i \}.$$ 

Theorem

- A strategy profile $s^*$ is a NE if and only if $\forall i \in N, s_i^* \in B_i(s_{-i}^*)$

Implication

- We can use the best response function to identify Nash Equilibria. Nash Equilibria occur where the best response functions for all the players intersect.
How do I find Nash equilibria?

Determine the best responses, that is the best strategy for a player given the strategies played by opponents. The best responses for each player intersect at the Nash equilibrium.

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<td>C</td>
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<tr>
<td>D</td>
<td>2, 0</td>
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</table>
Nash Equilibrium

Given column plays C, what is best response for Row?
Nash Equilibrium

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<th>D</th>
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<tbody>
<tr>
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<td>3, 3</td>
<td>0, 2</td>
</tr>
<tr>
<td>D</td>
<td>2, 0</td>
<td>1, 1</td>
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</tbody>
</table>

Given column plays C, what is best response for Row?

C because 3 > 2.
Given column plays C, what is best response for Row?

C because 3 > 2.

Let’s circle 3 because it indicates one of the best responses.
Given column plays D, what is best response for Row?

Nash Equilibrium

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<td>D</td>
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Nash Equilibrium

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<tr>
<td><strong>D</strong></td>
<td>2, 0</td>
<td>1, 1</td>
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</table>

Given column plays D, what is best response for Row?

D because 1 > 0.
Given column plays D, what is best response for Row?

D because $1 > 0$.

Let’s circle 1 because it indicates one of the best responses.
Given “Row” plays C, what is best response for Column?
Nash Equilibrium

<table>
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<tr>
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<th>Column</th>
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<tbody>
<tr>
<td><strong>C</strong></td>
<td>3</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>2, 0</td>
</tr>
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</table>

Given “Row” plays C, what is best response for Column?

C because 3 > 2.
Nash Equilibrium

Given “Row” plays C, what is best response for Column?

C because 3 > 2.

Let’s circle 3 because it indicates one of the best responses.
Nash Equilibrium

Given “Row” plays D, what is best response for Column?

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<th>Column</th>
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<tbody>
<tr>
<td></td>
<td>C</td>
</tr>
<tr>
<td>C</td>
<td>3, 3</td>
</tr>
<tr>
<td>D</td>
<td>2, 0</td>
</tr>
</tbody>
</table>
Given “Row” plays D, what is best response for Column?

D because 1 > 0.
Given “Row” plays D, what is best response for Column?

D because 1 > 0.

Let’s circle 1 because it indicates one of the best responses.
Nash Equilibrium

Where the best responses intersect \{C;C\} and \{D;D\} are Nash Equilibria.

N.E. = \{C;C\} and \{D;D\}

Note: equilibria are always stated in terms of strategies, never in terms of payoffs.
Practice: Chicken

What’s the Nash Equilibrium in this game? *Hint*: use best responses

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<td>2, 2</td>
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<tr>
<td>H</td>
<td>3, 1</td>
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</tbody>
</table>
### Practice: Nash Equilibrium

<table>
<thead>
<tr>
<th>Row</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>x</td>
<td>y</td>
<td>z</td>
</tr>
<tr>
<td>A</td>
<td>(2, 3)</td>
<td>(-16, 2)</td>
<td>(5, 0)</td>
</tr>
<tr>
<td>B</td>
<td>(5, 6)</td>
<td>(4, 6)</td>
<td>(6, 4)</td>
</tr>
<tr>
<td>C</td>
<td>(8, 0)</td>
<td>(3, 10)</td>
<td>(1, 8)</td>
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</tbody>
</table>
### Practice 2: Nash Equilibrium

<table>
<thead>
<tr>
<th>Row</th>
<th>Column</th>
<th>None</th>
<th>Some</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>None</td>
<td>0, 20</td>
<td>5, 15</td>
<td>10, 10</td>
</tr>
<tr>
<td></td>
<td>Some</td>
<td>1, 3</td>
<td>4, 4</td>
<td>15, 5</td>
</tr>
<tr>
<td></td>
<td>None</td>
<td>2, 2</td>
<td>3, 1</td>
<td>20, 0</td>
</tr>
</tbody>
</table>
Continuous Strategies

Rather than making a discrete choice to contribute a none, some, or all, we could consider making a continuous choice between 0 (none), 1 (all), and everything in between.

Continuous strategies are common in spatial voting models.
Exercise 42.2 (A Joint Project)

Description of the Game

• Two people are working on a joint project (like a group homework assignment). Each person must choose an effort level \( x \in [0, 1] \). Effort costs \( c(x_i) = x_i^2 \). The benefit of their efforts is \( f(x_1, x_2) = 3x_1x_2 \), which is split equally among them.

Utility Functions

• The utility for each individual is:

\[
u_i(x_i, x_j) = \frac{3}{2} x_i x_j - x_i^2\]
Exercise 42.2 (A Joint Project)

\[ u_i(x_i, x_j) = \frac{3}{2} x_i x_j - x_i^2 \]

Find Best Response Functions:

First, maximize player 1’s utility with respect to her effort:

\[ \frac{\partial u_i(x_i, x_j)}{\partial x_1} = \frac{3}{2} x_2 - 2x_1 \]

Solving the first order condition (FOC) for \( x_1 \) (i.e., setting the derivate equal to 0 and solving for \( x_1 \)) yields the best response for player 1 to player 2’s choice effort:

\[ x_1^* = B_1(x_2) = \frac{3}{4} x_2. \]

Symmetrically, the best response function for player 2 is:

\[ x_2^* = B_2(x_1) = \frac{3}{4} x_1. \]
Exercise 42.2 (A Joint Project)

Nash Equilibrium:

In equilibrium, both players must be playing a best response to the other player’s effort level. Mathematically, it must be that:

\[ x_1^* = \frac{3}{4} x_2^* \]
\[ x_2^* = \frac{3}{4} x_1^* \]

Substituting yields:

\[ x_2^* = \frac{9}{16} x_2^* \iff x_2^* = 0. \]

This implies \( x_1^* = 0. \)
Exercise 42.2 (A Joint Project)
Application: Electoral competition

Baseline model: Office-motivated candidates

• Hotelling (1929), Downs (1957)
• Two candidates choose policy platforms
• Platforms credibly translate into policy outcomes
• Candidates care only about winning
  – they prefer winning to tying to losing
• Voters
  – Care about policy outcomes
  – Single-peaked and symmetric preferences
  – Continuous distribution of voters with median m
  – Voters are non-strategic
Baseline model: Office Motivated Candidates

Players: Candidates, $N = \{1, 2\}$

Strategies: Platforms, $X_i = \mathbb{R}$.

Preferences

$$u_1(x_1, x_2) = \begin{cases} 
1 & \text{if } |x_1 - m| < |x_2 - m| \\
0 & \text{if } |x_1 - m| = |x_2 - m| \\
-1 & \text{if } |x_1 - m| > |x_2 - m| 
\end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} 
1 & \text{if } |x_1 - m| > |x_2 - m| \\
0 & \text{if } |x_1 - m| = |x_2 - m| \\
-1 & \text{if } |x_1 - m| < |x_2 - m| 
\end{cases}$$

*Note*: continuous strategies, but not continuous payoffs. Hence, we won’t use first derivatives.
Nash equilibrium

Proposition \((m, m)\) is the unique Nash equilibrium

Proof

Step 1. Show that \((m, m)\) is a NE
Nash equilibrium

Step 2. Show that no other \((x_1, x_2)\) is a NE
Best response functions for player 1

Case 1. $x_2 < m$

Case 2. $x_2 = m$

Case 3. $x_2 > m$
Summary

• Strategic games
  – Players
  – Strategies for each player
  – Preferences over strategy profiles

• Nash equilibrium
  – Strategy profile such that no player has a unilateral incentive to deviate
  – Best Response Functions (for both discrete and continuous strategies)
  – Predicts stable outcomes, but may not be unique