SPATIAL VOTING
(MULTIPLE DIMENSIONS)
Assumptions

• Alternatives are points in an n-dimensional space.
  – Examples for 2D:
    • Social Issues and Economic Issues
    • Domestic Spending and Foreign Spending

• Single-peaked preferences
  – Preferences are satiable
  – Each agent has an ideal point (most-preferred alternative)

• Symmetric preferences
  – Utility declines as a distance from ideal point increases
Single Dimension

Utility is a decreasing function of distance between the alternative and ideal point

Linear (absolute value)

\[ U(x) = -|x - 2| \]

Quadratic

\[ U(x) = -(x - 2)^2 \]
Two or more dimensions

Linear

$$U(x) = - \sum_{j=1}^{k} \alpha_j |x_j - \theta_j|$$

j indexes dimensions
$$\alpha_j = \text{weight on dimension } j$$
$$\theta_j = \text{ideal policy on dimension } j$$

Quadratic

$$U(x) = - \sum_{j=1}^{k} \alpha_j (x_j - \theta_j)^2$$

...let’s focus on this one.
Two dimensions

$$U(x, y) = -(x - \theta_1)^2 - (y - \theta_2)^2$$
Projection onto policy space

Indifference curve = set of points individual is indifferent between

Example

\[ U(x, y) = -(x - \theta_1)^2 - (y - \theta_2)^2 \]
Projection onto policy space

\[ w \in P \text{ z } P \times I \text{ y} \]
Example

Suppose $X = \mathbb{R}^2$ and consider single-peaked and symmetric preferences with an ideal point at (2,1). Rank order the following alternatives: (3,1), (0,0), (1,-1).

*Hint*: use the equation for a circle (two slides back).
Extra Credit HW 7
Effect of weights

Equal weights:
Indifference circle

\[ \alpha_1 = \alpha_2 \]

e.g., \( U(x, y) = -x^2 - y^2 \)

Symmetric

Different weights:
Indifference ellipse

\[ \alpha_1 < \alpha_2 \]

e.g., \( U(x, y) = -x^2 - 4y^2 \)

non-symmetric
Recall: the majority rule win set of $x$ is the set of alternatives that a majority prefers to $x$: $W(x) = \{y \mid yP x\}$

Created by drawing indifference curves through $x$ and shading points that a majority of voters prefer to $x$.  

Win Set
Win Set
Win Set

- $V_1$
- $V_2$
- $V_3$
Win Set

\[ V_1, V_2, V_3 \]
Win set of x
Win set of $y$
When is the core non-empty in two dimensions? (i.e. when does an equilibrium exist?)

Put Differently:
- What alternatives are in equilibrium? Under what conditions?
- What alternatives have completely empty win sets?
If $q = V2$, then there is no point that a majority prefer to $q$. 
When is the core non-empty in two dimensions?

Perhaps $q = V5$ is the core.
When does the core exist in two dimension?

Note: $q$ beats $x$. 

Cut line: perpendicular bisector between $q$ and $x$, demarcating those who prefer $x$ and those who prefer $q$. 

Note: $q$ beats $x$. 

When is the core non-empty in two dimensions?
When is the core non-empty in two dimensions?
When is the core non-empty in two dimensions?
Core is non-empty
(i.e., \( q = V5 \) is an equilibrium)
Core is non-empty
(i.e., $q=V5$ is an equilibrium)

Because $V1$ and $V3$ are always in opposition.
Core is non-empty
(i.e., $q=V5$ is an equilibrium)

and $V2$ and $V4$ are always in opposition.
Core is non-empty
(i.e., \( q = V5 \) is an equilibrium)

Which makes \( W(V5) = \emptyset \)
Ex 2: Core *is* empty (i.e., no equilibrium)
Ex 3: Core *is* empty
(i.e., no equilibrium)
Ex 4: Core *is* empty (i.e., no equilibrium)
Ex 2: Core *is* empty
(i.e., no equilibrium)
Ex 5 Core is empty (i.e., no equilibrium)
Plott conditions

• The **Plott conditions** are satisfied if and only if ideal points are distributed in a radially symmetric fashion around a policy $x^*$ and $x^*$ is a voter’s ideal point.

• **Radial symmetry** means that ideal points on opposite sides of $x^*$ can be paired by a line through $x^*$ and the lines defined by each pair intersect at $x^*$.

• The ideal point $x^*$ is sometimes called the **generalized median voter**.

**Proposition**  In the spatial model with Euclidean preferences and an odd number of voters, the majority rule core is $x^*$ if and only if the Plott conditions hold.
Plott conditions
Multidimensional Spatial Voting

Vote Cycle

1. Intransitive group preferences caused by the voting rule.
3. Note: the core would be empty if all of the alternatives were in a vote cycle.

```
    Rock
  /     \
Scissors Paper
```
Multidimensional Spatial Voting

Vote Cycle

1. Intransitive group preferences caused by the voting rule.
3. **Note**: the core would be empty if all of the alternatives were in a vote cycle.

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X vs Z:
Multidimensional Spatial Voting

Vote Cycle

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3. **Note**: the core would be empty if all of the alternatives were in a vote cycle.

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X vs Z: Z wins (2 to 1)
Multidimensional Spatial Voting

Vote Cycle

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X vs Z: Z wins (2 to 1)
Z vs Y: Y wins (2 to 1)
Multidimensional Spatial Voting

Vote Cycle

1. Intransitive group preferences caused by the voting rule.
3. Note: the core would be empty if all of the alternatives were in a vote cycle.

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- X vs Z: Z wins (2 to 1)
- Z vs Y: Y wins (2 to 1)
- Y vs X: X wins (2 to 1)
Multidimensional Spatial Voting

Vote Cycle

1. Intransitive group preferences caused by the voting rule.
3. Note: the core would be empty if all of the alternatives were in a vote cycle.

So we have demonstrated the possibility of majority rule vote cycles, but how general is the problem?

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X vs Z: Z wins (2 to 1)
Z vs Y: Y wins (2 to 1)
Y vs X: X wins (2 to 1)

Intransitivity
Top cycle set

Informally, the top cycle set is the set of alternatives $T$ such that $x \in T$ defeats any $y \notin T$ (directly or via a finite chain) but there is a preference cycle over the alternatives in $T$

Example

- aPb, bPc, cPa
- aPd, bPd, cPd
McKelvey’s Theorem

In the spatial model with Euclidean preferences, either the majority rule \textit{core is non-empty} (i.e. something is in equilibrium) or \textit{the top cycle set is the entire set of alternatives}

Corollaries
• If the Plott conditions do not hold, then the top cycle set is the entire set of alternatives.
• If the Plott conditions do not hold, then the majority rule core is empty (i.e. no point is in equilibrium).

Interpretation
• With complete information, an agenda setter “could” move us anywhere in the space.
Suppose the status quo is at $\times$ and the Yabloko gets to set the agenda.
McKelvey claimed that an agenda setter “could” move us anywhere in the space.
McKelvey claimed that an agenda setter “could” move us anywhere in the space.

Liberal-States: Liberal-States

Regions of Russia: Regions of Russia

Independent: Independent

Edinstvo: Edinstvo

OVR: OVR

SPS: SPS

Yabloko: Yabloko

Agora: Agora
...Even to the ideal point of the agenda setter.
We are going to construct an agenda starting with y and ending in x.
Constructing a cycle

Note that $x \notin W(y)$
Constructing a cycle
Constructing a cycle
Constructing a cycle
Constructing a cycle
Constructing a cycle
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Constructing a cycle
Not only could we construct an agenda from y to x, we can get back to y again (i.e. intransitivity).
Intuition: Shifting majorities

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Note that the cycle is explained by shifting coalitions.

Let’s follow the steps.
## Intuition: Shifting majorities

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### Intuition: Shifting majorities

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![Diagram](image.png)
# Intuition: Shifting majorities

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![Diagram showing the shifting of majorities]

65
**Intuition: Shifting majorities**

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![Diagram](image)
Remarks

• Even though the win set of a “centrally” located alternative is smaller than the win set of an “extreme” alternative, an agenda of pairwise majority rule votes can be constructed such that an extreme alternative emerges as the final outcome.

• Tips for constructing a cycle
  – Shift or alternate the majority coalition for each successive vote
  – Choose alternatives successively further away from the “middle,” thereby creating larger and larger win sets.
Substantive implications

Chaos?
• Common misinterpretation (1980s “new institutionalism”)
• McKelvey’s Theorem *does not* imply “chaos”

Agenda power?
• Suppose an individual has the power to choose the voting agenda, McKelvey’s Theorem *does imply* that the agenda setter is quite powerful
• From any status quo \( y \), a single agenda setter can produce *any* outcome \( x \) by choosing the appropriate agenda.
Summary

• Single dimension
  – The location of the median voter’s ideal point is the unique element of the majority rule core.
  – Majority rule creates a transitive preference order for society that is identical to the median voter’s preference order.

• Multiple dimensions. If the Plott conditions are not met,…
  – the majority rule core is empty (i.e., no equilibrium).
  – Majority rule creates an *intransitive* order for the entire set of alternatives. Hence, an agenda setter could take us anywhere.
Unanimity Rule vs Majority Rule

• Win Sets
  – Generally, the larger the supermajority rule threshold, the smaller the winset.

• Cores
  – Generally, the larger the supermajority rule threshold, the more likely there is a non-empty core (i.e. an equilibrium).
  – For unanimity rule, the set of points in the convex hull are in the core (i.e. they are in equilibrium).
Unanimity Rule vs Majority Rule

4 ideal points
Unanimity Rule vs Majority Rule

Status Quo (Q)
Unanimity Rule vs Majority Rule

What set of alternatives beat Q by Majority Rule?
What set of alternatives beat Q by Majority Rule?

Red petals
What set of alternatives beat Q by Unanimity Rule?

None
Unanimity Rule vs Majority Rule

New Q:
What set of alternatives beat Q by majority rule?
New Q:
What set of alternatives beat Q by majority rule?

Red petals
Unanimity Rule vs Majority Rule

What set of alternatives beat Q by unanimity rule?

none
In General:
Anything in the convex hull (the box around the ideal points) is in equilibrium using unanimity rule.
In General:

Anything in the convex hull (the box around the ideal points) is in equilibrium using unanimity rule.

Anything outside the hull is NOT in equilibrium using unanimity rule.
New Q:
What set of alternatives beat Q by majority rule?
New Q:
What set of alternatives beat Q by majority rule?
Red petals
What set of alternatives beat Q by unanimity rule?
Unanimity Rule vs Majority Rule

What set of alternatives beat Q by unanimity rule?
Red petals

Notice: the winset is smaller for unanimity rule than it was for majority rule.
New Q:
What set of alternatives beat Q by majority rule?
New Q:
What set of alternatives beat Q by majority rule?
Red petals
What set of alternatives beat Q by unanimity rule?
What set of alternatives beat Q by unanimity rule?

Red Petals

Again: the winset for unanimity rule is smaller than the winset for majority rule.

Is the same true in single dimensional models?

Yes.
Summary

• Unanimity Rule
  – The points inside the convex hull (i.e. the perimeter around the ideal points) are in the core (i.e. cannot be beaten).
  – The unanimity rule win set is a subset of the majority rule win set.

• Caplin and Nalebuff (1988), *Econometrica*
  – Any k-majority rule threshold greater that 64% of the chamber will not produce vote cycling.